

A PROOF ON HYPOTHESIS OF DIRICHLET DIVISOR PROBLEM

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Abstract

We have proved that $\Delta(X) = X^{\frac{1}{4}+\varepsilon(X)}$, $\varepsilon(X) = \frac{c}{\log \log X}$.

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1 RESULTS AND THOUGHT OF PROOF

Let

$$D(X) = \sum_{n \leq X} d(n), \quad (1.1)$$

where $d(n)$ denotes the number of divisors of n as usual, let

$$\Delta(X) = D(X) - X \log X - (2\gamma - 1)X, \quad (1.2)$$

and let $\theta = \inf\{\theta^* : \Delta(X) \ll X^{\theta^*}\}$. Then the following bounds for θ have been obtained:

- | | | | | |
|----|------------------------------|-------------------|----------------------------------|-------|
| 1. | $\theta \leq \frac{1}{2}$ | $= 0.50000 \dots$ | Dirichlet | 1849. |
| 2. | $\theta \leq \frac{1}{3}$ | $= 0.33333 \dots$ | Voronoi | 1903 |
| 3. | $\theta \leq \frac{33}{100}$ | $= 0.33000 \dots$ | Vander Corput[6] | 1922 |
| 4. | $\theta \leq \frac{27}{82}$ | $= 0.32926 \dots$ | Van der Corput[7] | 1928 |
| 5. | $\theta \leq \frac{15}{46}$ | $= 0.32608 \dots$ | Chih, T. T.[8](1950), Richert[9] | 1953 |
| 6. | $\theta \leq \frac{13}{40}$ | $= 0.32500 \dots$ | Yin Wenlen[10] | 1959 |

7.	$\theta \leq \frac{12}{37}$	$= 0.32432 \dots$	Kolesnik[11]	1969
8.	$\theta \leq \frac{346}{1067}$	$= 0.32427 \dots$	Kolesnik[12]	1973
9.	$\theta \leq \frac{35}{108}$	$= 0.32407 \dots$	Kolesnik[13]	1982
10.	$\theta \leq \frac{139}{429}$	$= 0.32400 \dots$	Kolesnik[14]	1985
11.	$\theta \leq \frac{7}{22}$	$= 0.31818 \dots$	Iwniec and Mozzochi[15]	1988
12.	$\theta \leq \frac{131}{416}$	$= 0.31490 \dots$	M.N.Huxley[16]	2003

The hypothesis value is $\theta = 1/4$. Moreover, in 1916, Hardy [15] proved that $\theta \geq 1/4$.

In this paper we are going to prove the hypothesis is true. For this purpose, we will prove the following theorem:

Theorem Let $X_1 < X_2$ be large positive number, $X_1 \leq X \leq X_2$ and

$$L = \log X_2, \quad X_1 \asymp X_2. \quad (1.3)$$

Then

$$\Delta(X) = X^{\frac{1}{4}} \Lambda(X) + \delta(X), \quad (1.4)$$

where $\delta(X) \ll X^\varepsilon$, $\varepsilon > 0$ arbitrary small, and

$$\begin{aligned} \Lambda(X) = & \frac{X}{(m_0 + 1)^2} \sum_{a=1}^{4K} C_1(a) \sum_{n \leq \frac{X_2 L^2}{V}} d(n) n^{-\frac{5}{4}} \cos(4\pi \sqrt{n}(\sqrt{X} + \frac{am_0}{2\sqrt{X}}) - \frac{3\pi}{4}) \\ & + \frac{\sqrt{X}}{m_0 + 1} \sum_{a=1}^{4K+2} C_2(a) \sum_{n \leq \frac{X_2 L^2}{V}} d(n) n^{-\frac{7}{4}} \cos(4\pi \sqrt{n}(\sqrt{X} + \frac{am_0}{2\sqrt{X}}) - \frac{5\pi}{4}), \end{aligned} \quad (1.5)$$

with

$$K = [\frac{C_0 L}{\log L}], \quad C_0 \geq 200, \quad (1.6)$$

$C_1(a), C_2(a)$ being independent of X , and

$$C_1(a), C_2(a) \ll \exp(O(\frac{L}{\log L})), \quad m_0 = 2[\sqrt{X_2} L^{-2}]. \quad (1.7)$$

Clearly, L , K and m_0 are independent of X by (1.3) and (1.6). The following Corollary is obtained immediately by this Theorem.

Corollary

$$\Delta(X) \ll X^{\frac{1}{4} + \varepsilon(X)}, \quad 0 < \varepsilon(X) = O(\frac{1}{\log \log X}). \quad (1.8)$$

Proof: In fact,

$$\Lambda(X) \ll \exp(O(\frac{L}{\log L})) (\sum_{n=1}^{\infty} d(n)n^{-\frac{5}{4}} + \sum_{n=1}^{\infty} d(n)n^{-\frac{7}{4}}) \ll \exp(O(\frac{L}{\log L})) \ll X^{O(\frac{1}{\log L})},$$

where the last inequality is given by $L \asymp \log X_1 \asymp \log X$. \square

Clearly, the hypothesis is true by this Corollary.

To prove Theorem 1.1, suppose that

$$U = [X], \tag{1.9}$$

$$A = C_0 \sqrt{UL}, \quad C_0 \geq 200, \tag{1.10}$$

$$B = \sqrt{U} + \frac{k+j}{2\sqrt{U}} + \eta, \tag{1.11}$$

$$k = 0, 1, \dots, m \ll m_0 = [\sqrt{X_2}L^{-2}], \tag{1.12}$$

$$j = 0, \pm 1, \dots, \pm j_0, \quad j_0 \leq \sqrt{V}L, \tag{1.13}$$

$$V = X_2^{\varepsilon_0}, \quad \varepsilon_0 = \frac{1}{\log L}, \tag{1.14}$$

$$\eta = \eta_1 + \dots + \eta_K, \quad K = [\frac{C_0 L}{\log L}], \tag{1.15}$$

$$\eta_\lambda = \frac{k_1}{\sqrt{U}}, \quad \lambda = 1, 2, \dots, K, \tag{1.16}$$

$$k_1 = 0, 1, \dots, m_1 \ll m_0 = [\sqrt{X_2}L^{-2}], \tag{1.17}$$

$$\xi_1 = \frac{1}{16\sqrt{U}}, \quad \xi_2 = \frac{3}{16\sqrt{U}}, \tag{1.18}$$

$$S(B) = A \int_{-\frac{5\sqrt{L}}{A}}^{\frac{5\sqrt{L}}{A}} e^{-\pi A^2 \theta^2} d\theta \sum_{\xi_1 < \sqrt{n-B-\theta} < \xi_2} d(n)n^{-\frac{1}{4}}. \tag{1.19}$$

Clearly,

$$\begin{aligned} (B + \theta + \xi_2)^2 - (B + \theta + \xi_1)^2 &= (\xi_2 - \xi_1)(2B + 2\theta + \xi_1 + \xi_2) \\ &= \frac{1}{8\sqrt{U}}(2\sqrt{U} + O(1)) \\ &< \frac{1}{3} \end{aligned} \tag{1.20}$$

and

$$\begin{aligned}
\{(B + \theta + \xi)^2\} &= \{(\sqrt{U} + \frac{k+j}{2\sqrt{U}} + \eta + \theta + \xi)^2\} \\
&= \{U + k + j + 2\sqrt{U}\eta + 2\sqrt{U}\xi + 2\sqrt{U}\theta + (\frac{k+j}{2\sqrt{U}} + \eta + \theta + \xi)^2\} \\
&= \{2\sqrt{U}\xi + 2\sqrt{U}\theta + O(L^{-1})\}.
\end{aligned}$$

By (1.15) and (1.16) we know that $2\sqrt{U}\eta$ is an integer, thus

$$2\sqrt{U}(\xi + \theta) \geq 2\sqrt{U}(\frac{1}{16\sqrt{U}} - \frac{5\sqrt{L}}{A}) = \frac{1}{8} - \frac{10}{C_0} \geq \frac{1}{8} - \frac{1}{20} = \frac{3}{40}$$

and

$$2\sqrt{U}(\xi + \theta) \leq 2\sqrt{U}(\frac{3}{16\sqrt{U}} + \frac{5\sqrt{L}}{A}) = \frac{3}{8} + \frac{10}{C_0} \leq \frac{3}{8} + \frac{1}{20} = \frac{17}{40},$$

therefore,

$$\frac{1}{20} < \{(B + \theta + \xi)^2\} < \frac{9}{20} \quad (1.21)$$

for $\frac{1}{16\sqrt{U}} \leq \xi \leq \frac{3}{16\sqrt{U}}$, $-\frac{5\sqrt{L}}{A} \leq \theta \leq \frac{5\sqrt{L}}{A}$. It follows from (1.20) and (1.21) that there doesn't exist any integer in the interval

$$[(B + \theta + \xi_1)^2, (B + \theta + \xi_2)^2].$$

Hence

$$\sum_{\xi_1 < \sqrt{n} - B - \theta \leq \xi_2} = \sum_{(B + \theta + \xi_1)^2 < n \leq (B + \theta + \xi_2)^2} = 0, \quad (1.22)$$

and by (1.19),

$$S(B) = 0. \quad (1.23)$$

Denote

$$f(\eta) \Big|_{\eta_K} = f(\eta_1 + \eta_2 + \cdots + \eta_K) \Big|_{\eta_1=0}^{\eta_1=\frac{k_1}{\sqrt{U}}} \cdots \Big|_{\eta_K=0}^{\eta_K=\frac{k_1}{\sqrt{U}}}. \quad (1.24)$$

By (1.23),

$$S(B) \Big|_{\eta_K} = 0. \quad (1.25)$$

Therefore,

$$\Omega = \frac{1}{\sqrt{V}} \sum_{-\sqrt{V}L \leq j \leq \sqrt{V}L} e^{\frac{-\pi j^2}{V}} \sum_{k_1=m_0+1}^{2m_0} \sum_{m=m_0+1}^{2m_0} \sum_{k=0}^m S(B) \Big|_{\eta_K} = 0. \quad (1.26)$$

On the other hand, in the following we are going to prove that

$$\Omega = \frac{(-1)^{K+1}(m_0+1)^2}{4X^{\frac{1}{4}}}(\Delta(X) - X^{\frac{1}{4}}\Lambda(X) + O(X^\varepsilon)). \quad (1.27)$$

This and (1.26) lead to (1.4).

For convenience in the proof we are going to use the following definition.

Definition 1.1 If

$$f(\eta) \Big|_{\eta_K} = f(\eta_1 + \eta_2 + \cdots + \eta_K) \Big|_{\eta_1=0}^{\eta_1=\frac{k_1}{\sqrt{U}}} \cdots \Big|_{\eta_K=0}^{\eta_K=\frac{k_1}{\sqrt{U}}} \ll Y,$$

then we write

$$f(\eta) = O_\eta(Y), \text{ or } f(\eta) \ll_\eta Y. \quad (1.28)$$

Moreover, we denote

$$f(\eta) \Big|_{\eta_K} = \sum_{\eta_1=0, \frac{k_1}{\sqrt{U}}} \cdots \sum_{\eta_1=0, \frac{k_K}{\sqrt{U}}} \left| f(\eta_1 + \eta_2 + \cdots + \eta_K) \right|.$$

Clearly,

$$f(\eta) \Big|_{\eta_K} \leq f(\eta) \Big|_{\eta_K}. \quad (1.29)$$

Particularly, if $f(\eta) \ll M$, then

$$f(\eta) \Big|_{\eta_K} \leq f(\eta) \Big|_{\eta_K} \ll M 2^K. \quad (1.30)$$

For arbitrary small $\varepsilon > 0$, or $\varepsilon = \varepsilon(U) = O(1/\log L)$, we are going to use expressions, such as

$$U^{2\varepsilon} \ll U^\varepsilon, \quad (1.31)$$

without explanation.

2 SOME LEMMAS (I)

Let

$$S' = \sum_{a \leq n \leq b} 'd(n)f(n),$$

where $0 < a < b < \infty$, $f(x) \in C^2[a, b]$, \sum' is that if a or b is an integer, then $\frac{1}{2}d(a)n(a)$ or $\frac{1}{2}d(b)n(b)$ is to be counted instead of $d(a)n(a)$ or $d(b)n(b)$, then (for example, (3.2), (3.15) in [2])

$$\begin{aligned} S' &= \sum_{a \leq n \leq b} 'd(n)f(n) \\ &= \int_a^b (\log x + 2\gamma)f(x)dx + \sum_{n=1}^{\infty} d(n) \int_b^a f(x)(\Theta(nx) + O((nx)^{-\frac{5}{4}}))dx, \end{aligned} \quad (2.1)$$

where

$$\Theta(nx) = \sqrt{2}(nx)^{-1/4}(\cos(4\pi\sqrt{nx} + \pi/4) - \frac{1}{32\pi\sqrt{nx}}\cos(4\pi\sqrt{nx} + 3\pi/4)). \quad (2.2)$$

In the following we are going to evaluate general sum $\sum d_\lambda(n)f(n)$ in Γ -function with more accurate O -term, and indicate the results in the finite term. Denote

$$S_\lambda = \frac{1}{h_0^{\lambda-1}} \int_0^{h_0} \cdots \int_0^{h_0} dh_1 \cdots dh_{\lambda-1} \sum_{a+h < n^{\frac{1}{\lambda}} \leq b+h} d_\lambda(n)f(n), \quad (2.3)$$

where

$$d_\lambda(n) = \sum_{n_1 \cdots n_\lambda = n} 1, \quad \lambda \geq 2, \quad (2.4)$$

$$f(x) \in C^\lambda[a^\lambda, (b + (\lambda - 1)h_0)^\lambda], \quad 0 < h_0 < \frac{a}{2}, \quad (2.5)$$

$$h = h_1 + \cdots + h_{\lambda-1}.$$

Lemma 2.1 *Let*

$$\beta_0 = \sum_{j=1}^{\lambda} \binom{\lambda}{j} ((\lambda - 1)h_0)^j b^{\lambda-j},$$

if h_0 is sufficient small such that

$$\beta_0 < \min(1 - \{a^\lambda\}, 1 - \{b^\lambda\}),$$

then

$$S_\lambda = \sum_{a < n^{\frac{1}{\lambda}} \leq b} d_\lambda(n)f(n). \quad (2.6)$$

In fact, in this case, there are not any integer in $(a^\lambda, (a + h)^\lambda]$ and $(b^\lambda, (b + h)^\lambda]$, hence,

$$\sum_{a^\lambda < n \leq (a+h)^\lambda} = 0, \quad \sum_{b^\lambda < n \leq (b+h)^\lambda} = 0.$$

Lemma 2.2 *For $0.9 < a < b < \infty$,*

$$S_\lambda = S_{\lambda l} + \sum_{n \leq N} \frac{d_\lambda(n)}{h_0^{\lambda-1}} \int_0^{h_0} \cdots \int_0^{h_0} dh_1 \cdots dh_{\lambda-1} \int_{(a+h)^\lambda}^{(b+h)^\lambda} f(x)\Theta_\lambda(nx)dx + O(\varepsilon_1) \quad (2.7)$$

holds for any $\varepsilon_1 > 0$ as $N \geq N(\varepsilon_1)$, where

$$S_{\lambda l} = \frac{1}{h_0^{\lambda-1}} \int_0^{h_0} \cdots \int_0^{h_0} dh_1 \cdots dh_{\lambda-1} \int_{(a+h)^\lambda}^{(b+h)^\lambda} f(n)p_\lambda(x)dx, \quad (2.8)$$

with

$$p_\lambda(x) = \frac{1}{2\pi i} \int_{|s|=1/2} x^s \zeta^\lambda(s+1) ds = \frac{1}{(\lambda-1)!} (\log x)^{\lambda-1} + C_1 (\log x)^{\lambda-2} + \dots + C_\lambda,$$

$$\begin{aligned} \Theta_\lambda(nx) &= \frac{2}{\sqrt{\lambda}} (nx)^{-\frac{1}{2} + \frac{1}{2\lambda}} \sum_{\alpha=0}^{\alpha_\lambda} \frac{\gamma_\alpha^{(\lambda)}(nx)^{\frac{-\alpha}{\lambda}}}{(2\pi\lambda)^\alpha} \cos(2n\lambda(nx)^{\frac{1}{\lambda}} + \frac{(\lambda-1)\pi}{4} + \frac{\alpha\pi}{2}) \\ &\quad + O((nx)^{\frac{-1}{2} + \frac{1}{2\lambda} - \frac{\alpha_\lambda+1}{\lambda}}) \end{aligned} \quad (2.9)$$

and

$$\gamma_0^{(\lambda)} = 1, \quad \gamma_1^{(\lambda)} = \frac{1-\lambda^2}{12},$$

for any natural number α_λ .

(In this paper we need the case $\lambda = 2$ only.)

Lemma 2.3

$$\begin{aligned} S &= S_2 = \frac{1}{h_0} \int_0^{h_0} dh \sum_{a+h < \sqrt{n} \leq b+h} d(n) f(n) \\ &= S_l + \sum_{n \leq N} \frac{d(n)}{h_0} \int_0^{h_0} dh \int_{(a+h)^2}^{(b+h)^2} f(x) \Theta(nx) dx + O(\varepsilon_1) \end{aligned} \quad (2.10)$$

holds for any $\varepsilon_1 > 0$ as $N \geq N(\varepsilon_1)$, where

$$0.9 < a < b < \infty, \quad 0 < h_0 < a/2, \quad f(x) \in C^2[a, b+h_0],$$

$$S_l = \frac{1}{h_0} \int_0^{h_0} dh \int_{(a+h)^2}^{(b+h)^2} f(x) (\log x + 2\gamma) dx, \quad (2.11)$$

$$\Theta(nx) = \sqrt{2} (nx)^{-\frac{1}{4}} \sum_{\alpha=0}^{\alpha_0} \frac{\gamma_\alpha (nx)^{\frac{-\alpha}{2}}}{(4\pi)^\alpha} \cos(4\pi\sqrt{nx} + \frac{\pi}{4} + \frac{\alpha\pi}{2}) + O((nx)^{\frac{-3}{4} - \frac{\alpha_0}{2}}), \quad (2.12)$$

with

$$\gamma_0 = 1, \quad \gamma_1 = -\frac{1}{8}, \quad \gamma_2 = \frac{9}{128}, \quad \gamma_3 = -\frac{75}{1024}, \quad \gamma_4 = \frac{3675}{2^{15}}, \quad \gamma_5 = -\frac{59535}{2^{18}}, \quad \gamma_6 = \frac{2401245}{2^{22}} \dots \quad (2.13)$$

for any natural number α_0 .

(in this paper we need not the concrete number values of γ_j , $j \geq 1$, we see γ_0, γ_1 be consistent for known results in (2.2).)

A corollary of Lemma 2.3 is the following Lemma 2.4

Lemma 2.4 *Let*

$$S = \frac{1}{h_0} \int_{h_1}^{h_2} g(h) dh \sum_{a+h < \sqrt{n} \leq b+h} d(n)f(n),$$

where $a + h_1 > 0.9$, $h_1 < h_2$, $0 < a < b < \infty$, $g(h) \in C^2[h_1, h_2]$, $f(x) \in C^2[(a + h_1)^2, (b + h_2)^2]$, $h_0 = \int_{h_1}^{h_2} g(h) dh > 0$, then

$$\begin{aligned} S &= \frac{1}{h_0} \int_{h_1}^{h_2} g(h) dh \int_{(a+h)^2}^{(b+h)^2} f(x)(\log x + 2\gamma) dx \\ &\quad + \sum_{n \leq N} \frac{d(n)}{h_0} \int_{h_1}^{h_2} g(h) dh \int_{(a+h)^2}^{(b+h)^2} f(x)\Theta(nx) dx + O(\varepsilon_1) \end{aligned}$$

holds for any $\varepsilon_1 > 0$ as $N \geq N(\varepsilon_1)$, where $\Theta(nx)$ is (2.12).

Clearly, Lemma 2.2 has also similar corollary.

Proof of Lemma 2.2: Let

$$D_\lambda(x) = \sum_{n \leq x} d_\lambda(n), \quad (2.14)$$

then

$$\begin{aligned} \sum_{(a+h)^\lambda < n \leq (b+h)^\lambda} d_\lambda(n)f(n) &= \int_{(a+h)^\lambda}^{(b+h)^\lambda} f(x) dD_\lambda(x) \\ &= f(x)D_\lambda(x) \Big|_{x=(a+h)^\lambda}^{x=(b+h)^\lambda} - \int_{(a+h)^\lambda}^{(b+h)^\lambda} f'(x)D_\lambda(x) dx. \end{aligned} \quad (2.15)$$

the last integral is

$$\begin{aligned} - \int_{(a+h)^\lambda}^{(b+h)^\lambda} f'(x)D_\lambda(x) dx &= - \int_{(a+h)^\lambda}^{(b+h)^\lambda} f'(x) \left(\int_0^x D_\lambda(y) dy \right)'_x dx \\ &= -f'(x) \int_0^x D_\lambda(y) dy \Big|_{x=(a+h)^\lambda}^{x=(b+h)^\lambda} + \int_{(a+h)^\lambda}^{(b+h)^\lambda} f''(x) \left(\int_0^x D_\lambda(y) dy \right) dx \\ &= \frac{-f'(x)}{2\pi i} \int_{\sigma_0 - iT}^{\sigma_0 + iT} \frac{x^{s+1} \zeta^\lambda(s) ds}{s(s+1)} \Big|_{x=(a+h)^\lambda}^{x=(b+h)^\lambda} \\ &\quad + \frac{1}{2\pi i} \int_{\sigma_0 - iT}^{\sigma_0 + iT} ds \int_{(a+h)^\lambda}^{(b+h)^\lambda} \frac{f''(x) x^{s+1} \zeta^\lambda(s)}{s(s+1)} dx + \delta, \end{aligned} \quad (2.16)$$

where $\sigma_0 = 1.1$ and

$$\begin{aligned}
\delta &= \frac{1}{2\pi i} \left(\int_{\sigma_0-i\infty}^{\sigma_0-iT} ds + \int_{\sigma_0+iT}^{\sigma_0+i\infty} ds \right) \left(-\frac{f'(x)x^{s+1}\zeta^\lambda(s)}{s(s+1)} \Big|_{x=(a+h)^\lambda}^{x=(b+h)^\lambda} \right. \\
&\quad \left. + \int_{(a+h)^\lambda}^{(b+h)^\lambda} \frac{f''(x)x^{s+1}\zeta^\lambda(s)}{s(s+1)} dx \right) \\
&\ll \int_T^\infty \frac{dt}{t^2} \ll \frac{1}{T}.
\end{aligned}$$

By (2.16),

$$-\int_{(a+h)^\lambda}^{(b+h)^\lambda} f' D_\lambda(x) dx = -\frac{1}{2\pi i} \int_{(a+h)^\lambda}^{(b+h)^\lambda} f' \int_{\sigma_0-iT}^{\sigma_0+iT} \frac{x^s \zeta^\lambda(s) ds}{s} + O\left(\frac{1}{T}\right). \quad (2.17)$$

Suppose e is the set of $h_{\lambda-1}$ and e satisfies $|x_\lambda + h_{\lambda-1} - m^{\frac{1}{\lambda}}| < \mu$, where m is an natural number, $x_\lambda = x + h_1 + \cdots + h_{\lambda-2}$ fixed, $E = (0, h_0) \setminus e$. We have

$$\begin{aligned}
&\int_e f((x_\lambda + h_{\lambda-1})^\lambda) D_\lambda((x_\lambda + h_{\lambda-1})^\lambda) dh_{\lambda-1} \ll \mu, \\
&\int_E f((x_\lambda + h_{\lambda-1})^\lambda) D_\lambda((x_\lambda + h_{\lambda-1})^\lambda) dh_{\lambda-1} \\
&= \int_E \frac{f((x_\lambda + h_{\lambda-1})^\lambda)}{2\pi i} dh_{\lambda-1} \int_{\sigma_0-iT}^{\sigma_0+iT} \frac{(x_\lambda + h_{\lambda-1})^{\lambda s} \zeta^\lambda(s)}{s} ds + \delta_E,
\end{aligned}$$

where

$$\begin{aligned}
\delta_E &= \int_E \frac{f((x_\lambda + h_{\lambda-1})^\lambda)}{2\pi i} dh_{\lambda-1} \left(\int_{\sigma_0-i\infty}^{\sigma_0-iT} + \int_{\sigma_0+iT}^{\sigma_0+i\infty} \right) \frac{(x_\lambda + h_{\lambda-1})^{\lambda s} \zeta^\lambda(s) ds}{s} \\
&= \int_E \frac{f((x_\lambda + h_{\lambda-1})^\lambda)}{2\pi i} dh_{\lambda-1} \sum_{n=1}^\infty d_\lambda(n) \left(\int_{\sigma_0-i\infty}^{\sigma_0-iT} + \int_{\sigma_0+iT}^{\sigma_0+i\infty} \right) \frac{1}{s} \left(\frac{x_\lambda + h_{\lambda-1}}{n^{\frac{1}{\lambda}}} \right)^{\lambda s} ds \\
&\ll \frac{1}{T} \int_E dh_{\lambda-1} \sum_{n=1}^\infty d_\lambda(n) \left(\frac{(x_\lambda + h_{\lambda-1})^\lambda}{n} \right)^{\sigma_0} \frac{1}{\left| \log \frac{x_\lambda + h_{\lambda-1}}{n^{\frac{1}{\lambda}}} \right|} \\
&\ll \frac{1}{T} \int_E dh_{\lambda-1} \left(1 + \sum_{\frac{2}{3} \leq |n^{\frac{1}{\lambda}} - x_\lambda - h_{\lambda-1}| \leq \frac{3}{2}} \frac{1}{|n^{\frac{1}{\lambda}} - x_\lambda - h_{\lambda-1}|} \right) \\
&\ll \frac{1}{T} \left(1 + \sum_{\frac{1}{12} \leq |n^{\frac{1}{\lambda}} - x_\lambda| \leq 3} \int_E \frac{dh_\lambda}{|n^{\frac{1}{\lambda}} - x_\lambda - h_{\lambda-1}|} \right) \\
&\ll \frac{1}{T} \log \frac{1}{\mu}
\end{aligned}$$

for $0 < h_0 < a/2$, $0 < \mu < 1/2$, and the constant in " O " is independent of T . Furthermore,

$$\int_e f((x_\lambda + h_{\lambda-1})^\lambda) dh_{\lambda-1} \int_{\sigma_0 - iT}^{\sigma_0 + iT} \frac{(x_\lambda + h_{\lambda-1})^\lambda \varsigma^\lambda(s)}{s} dS \ll \int_e \log T dh_{\lambda-1} \ll \mu \log T.$$

Hence, taking $\mu = \frac{1}{T}$, we have

$$\begin{aligned} & \int_0^{h_0} \cdots \int_0^{h_0} dh_1 \cdots dh_{\lambda-1} f(x) D_\lambda(x) \Big|_{x=(a+h)^\lambda}^{x=(b+h)^\lambda} = \\ & = \int_0^{h_0} \cdots \int_0^{h_0} dh_1 \cdots dh_{\lambda-1} f(x) \frac{1}{2\pi i} \int_{\sigma_0 - iT}^{\sigma_0 + iT} \frac{f(x) x^s \varsigma(s) ds}{s} \Big|_{x=(a+h)^\lambda}^{x=(b+h)^\lambda} + O\left(\frac{\log T}{T}\right). \end{aligned} \quad (2.18)$$

It follows from (2.3), (2.15), (2.17) and (2.18) that

$$\begin{aligned} S_\lambda &= \frac{1}{h_0^{\lambda-1}} \int_0^{h_0} \cdots \int_0^{h_0} dh_1 \cdots dh_{\lambda-1} \frac{1}{2\pi i} \int_{\sigma_0 - iT}^{\sigma_0 + iT} \frac{\varsigma^\lambda(s) ds}{s} \left(- \int_{(a+h)^\lambda}^{(b+h)^\lambda} f'(x) x^s dx \right. \\ & \quad \left. + f(x) x^s \Big|_{x=(a+h)^\lambda}^{x=(b+h)^\lambda} \right) + O\left(\frac{\log T}{T}\right) \\ &= \frac{1}{h_0^{\lambda-1}} \int_0^{h_0} \cdots \int_0^{h_0} dh_1 \cdots dh_{\lambda-1} \frac{1}{2\pi i} \int_{(a+h)^\lambda}^{(b+h)^\lambda} f(x) dx \int_{\sigma_0 - iT}^{\sigma_0 + iT} x^{s-1} \varsigma^\lambda(s) ds + O\left(\frac{\log T}{T}\right) \\ &= \frac{1}{h_0^{\lambda-1}} \int_0^{h_0} \cdots \int_0^{h_0} dh_1 \cdots dh_{\lambda-1} \frac{1}{2\pi i} \int_{(a+h)^\lambda}^{(b+h)^\lambda} f(x) dx \int_{1-\sigma_0 - iT}^{1-\sigma_0 + iT} x^{s-1} \varsigma^\lambda(s) ds \\ & \quad + S_\lambda + \delta_+ + \delta_- + O(\varepsilon_1) \end{aligned} \quad (2.19)$$

holds for any $\varepsilon_1 > 0$, as T large, where $h = h_1 + \cdots + h_{\lambda-1}$, $\lambda \geq 2$, S_λ is (2.8) and

$$\begin{aligned} \delta_\pm &= \mp \frac{1}{2\pi i h_0^{\lambda-1}} \int_{1-\sigma_0 \pm iT}^{\sigma_0 \pm iT} \varsigma^\lambda(s) I_\delta(s) ds, \\ I_\delta(s) &= \int_0^{h_0} \cdots \int_0^{h_0} dh_1 \cdots dh_{\lambda-1} \int_{a^\lambda}^{b^\lambda} f(x+h) (x+h)^{s-1} dx. \end{aligned} \quad (2.20)$$

Using the integrations by parts and (2.5), we see that $I_\delta \ll \frac{1}{T^\lambda}$. It is known that (for example, 2.1.8, 2.1.3 in [1])

$$\varsigma(s) = \frac{\varsigma(1-s)}{\chi(1-s)}, \quad \chi(s)\chi(1-s) = 1, \quad (2.21)$$

$$\frac{1}{\chi(s)} = 2(2\pi)^{-s} \Gamma(s) \cos \frac{\pi s}{2} = \left(\frac{t}{2\pi}\right)^{\sigma-1/2} e^{it \log \frac{t}{2\pi e}} \left(1 + O\left(\frac{1}{t}\right)\right), \quad (2.22)$$

$$\begin{aligned} & t > 1, \quad s = \sigma + it, \\ & \varsigma(s) \ll |t|^{\frac{1}{6}}, \quad \sigma \geq 1/2, |t| > 1. \end{aligned} \quad (2.23)$$

Hence

$$\begin{aligned}
\delta_{\pm} &\ll \frac{1}{T^{\lambda}} \left(\int_{1-\sigma_0}^{\frac{1}{2}} + \int_{\frac{1}{2}}^{\sigma_0} \right) |\varsigma^{\lambda}(\sigma \pm iT)| d\sigma \\
&\ll \frac{1}{T^{\lambda}} \int_{\frac{1}{2}}^{\sigma_0} (1 + T^{\lambda(\sigma-1/2)}) T^{\frac{\lambda}{6}} d\sigma \ll T^{\lambda(\frac{1}{6}+\sigma_0-3/2)} \ll T^{-\frac{\lambda}{5}}
\end{aligned} \tag{2.24}$$

for $\sigma_0 = 1.1$. Putting $s = 1 - s'$ in the integral (2.19), noticing (2.21) and (2.24),

$$\begin{aligned}
S_{\lambda} &= \frac{1}{h_0^{\lambda-1}} \int_0^{h_0} \cdots \int_0^{h_0} dh_1 \cdots dh_{\lambda-1} \frac{1}{2\pi i} \int_{(a+h)^{\lambda}}^{(b+h)^{\lambda}} \frac{f(x)dx}{2\pi i} \int_{\sigma_0-iT}^{\sigma_0+iT} \frac{x^{-s} \varsigma^{\lambda}(s) ds}{\chi^{\lambda}(s)} \\
&\quad + S_{\lambda l} + O(\varepsilon_1) \\
&= \sum_{n \leq N} \frac{d_{\lambda}(n)}{h_0^{\lambda-1}} \int_0^{h_0} \cdots \int_0^{h_0} dh_1 \cdots dh_{\lambda-1} \frac{1}{2\pi i} \int_{(a+h)^{\lambda}}^{(b+h)^{\lambda}} \frac{f(x)dx}{2\pi i} \int_{\sigma_0-iT}^{\sigma_0+iT} \frac{(nx)^{-s} ds}{\chi^{\lambda}(s)} \\
&\quad + S_{\lambda \delta} + \delta_N + O(\varepsilon_1),
\end{aligned} \tag{2.25}$$

where

$$\begin{aligned}
\delta_N &\ll \sum_{n > N} d_{\lambda}(n) \int_0^{h_0} \cdots \int_0^{h_0} dh_1 \cdots dh_{\lambda-1} \int_{(a+h)^{\lambda}}^{(b+h)^{\lambda}} |f(x)| \int_{-T}^T \frac{(n\chi)^{-\sigma_0} dt}{|\chi^{\lambda}(\sigma_0 + it)|} \\
&\ll \sum_{n > N} d_{\lambda}(n) n^{-1.1} \int_{-T}^T (|t| + 1)^{\lambda(1.1-1/2)} dt \ll \varepsilon_1
\end{aligned} \tag{2.26}$$

holds for any fixed T as N sufficient large.

Now estimate the integral

$$\delta_{\lambda}(n) = \int_{\sigma_0+iT}^{\sigma_0+iT_1} \frac{j_{\lambda}(s) n^{-s} ds}{\chi^{\lambda}(s)}, \tag{2.27}$$

$$j_{\lambda}(s) = \int_0^{h_0} \cdots \int_0^{h_0} dh_1 \cdots dh_{\lambda-1} \int_{(a+h)^{\lambda}}^{(b+h)^{\lambda}} f(x) x^{-s} dx \tag{2.28}$$

for $\sigma_0 = 1.1$, $T_1 \rightarrow \infty$, $n \leq N$. Using the integrations by parts it is easy to obtain that $j_{\lambda}(s) \ll \frac{1}{t^{\lambda}}$. Thus, by (2.22),

$$\delta_{\lambda}(n) \leq n^{-1.1} \int_T^{T_1} t^{\lambda(1.1-1/2)-\lambda} dt \leq n^{-1.1} \int_T^{T_1} t^{-0.4\lambda} dt \leq n^{-1.1} T^{1-0.4\lambda} \leq T^{-0.2} n^{-1.1} \tag{2.29}$$

for $\lambda \geq 3$. If $\lambda = 2$, then

$$\begin{aligned}
j_2(s) &= \int_0^{h_0} dh \int_{(a+h)^2}^{(b+h)^2} f(x) x^{-s} dx \\
&= \frac{2}{-2s+2} \int_0^{h_0} dh f((x+h)^2) (x+h)^{-2s+2} \Big|_{x=a}^{x=b} \\
&\quad - \frac{2}{-2s+2} \int_0^{h_0} dh \int_a^b (f((x+h)^2))' (x+h)^{-2s+2} dx \\
&= \frac{c_1}{t^2} f((x+h)^2) (x+h)^{-2s+3} \Big|_{x=a}^{x=b} \Big|_{h=0}^{h=h_0} \\
&\quad + \frac{c_2}{t^2} \int_0^{h_0} dh (f((x+h)^2))' (x+h)^{-2s+3} \Big|_{x=a}^{x=b} \\
&\quad + \frac{c_3}{t^2} \int_0^{h_0} dh (f((x+h)^2))'' (x+h)^{-2s+3} dx + O(t^{-3}). \tag{2.30}
\end{aligned}$$

Consider the integral

$$I_{2n}(T, T_1) = \int_T^{T_1} \frac{(n(x+h))^{-it} dt}{t^2 \chi^2(1.1+it)} = \int_T^{T_1} t^{-0.8} e^{2it \log \frac{t}{eM_n}} dt + O(T^{-0.8}), \tag{2.31}$$

where the last equation is given by (2.22), $\sigma_0 = 1.1$, and

$$M_n = 2\pi \sqrt{n(x+h)}. \tag{2.32}$$

If $M_n > T - \sqrt{T}$, then

$$\begin{aligned}
I_{2n}(T, T_1) &= \left(\int_{|t-M_n| \leq \sqrt{M_n}, T \leq t \leq T_1} + \int_{|t-M_n| > \sqrt{M_n}, T \leq t \leq T_1} \right) t^{-0.8} e^{2it \log \frac{t}{eM_n}} dt + O(T^{-0.8}) \\
&\ll M_n^{-0.8+1/2} + \frac{T_1^{-0.8}}{|\log \frac{T_1}{M_n}|} + \delta_T(M_n) + \frac{M_n^{-0.8}}{|\log \frac{M_n + \sqrt{M_n}}{M_n}|} + \frac{M_n^{-0.8}}{|\log \frac{M_n - \sqrt{M_n}}{M_n}|} \\
&\quad + \int_{|t-M_n| > M_n, T \leq t \leq T_1} \left(\frac{t^{-0.8-1}}{|\log \frac{t}{M_n}|} + \frac{t^{-0.8-1}}{|\log \frac{t}{M_n}|^2} \right) dt.
\end{aligned}$$

For any fixed M_n , as $T_1 \rightarrow \infty$, the second term is $\ll T_1^{-0.8} \ll T^{-0.8}$. If $M_n \leq \sqrt{T} + T$, then $\delta_T(M_n)$ is the first term. If $M_n > \sqrt{T} + T$, then

$$\delta_T(M_n) \ll \frac{T^{-0.8}}{|\log \frac{T}{M_n}|} \ll T^{-0.8+1/2}.$$

Thus,

$$\begin{aligned}
I_{2n}(T, T_1) &\ll T^{-0.8} + M_n^{-0.8+1/2} + \int_T^{T_1} t^{-0.8-1} dt \\
&\quad + \int_{\sqrt{M_n}/2 \leq |t-M_n| \leq 2M_n} \left(\frac{M_n}{|t-M_n|} + \frac{M_n^2}{|t-M_n|^2} \right) dt \\
&\ll M_n^{-0.8+1/2} + T^{-0.8} \ll T^{-0.3},
\end{aligned}$$

for $M_n > T - \sqrt{T}$. If $M_n \leq T - \sqrt{T}$, then it is easy to show $I_{2n}(T, T_1) \ll T^{-0.3}$. It follows from (2.27), (2.30) and the definition of $I_{2n}(T, T_1)$ that $\delta_2(n) \ll n^{-1.1}T^{-0.3}$. This and (2.29) give

$$\delta_\lambda(n) \ll n^{-1.1}T^{-0.2} \ll \varepsilon_1 n^{-1.1}, \quad \lambda \geq 2. \quad (2.33)$$

In the same way,

$$\int_{\sigma_0-iT_1}^{\sigma_0-iT} \frac{j_\lambda(s)n^{-s}ds}{\chi^\lambda(s)} \ll n^{-1.1}T^{-0.2} \ll \varepsilon_1 n^{-1.1}, \quad \lambda \geq 2, \quad \sigma_0 = 1.1. \quad (2.34)$$

Therefore

$$\lim_{T_1 \rightarrow \infty} \sum_{n \leq N} \frac{d_\lambda(n)}{h_0^{\lambda-1}} \int_0^{h_0} \cdots \int_0^{h_0} dh_1 \cdots dh_{\lambda-1} \int_{(a+h)^\lambda}^{(b+h)^\lambda} \left(\int_{\sigma_0+iT}^{\sigma_0+iT_1} + \int_{\sigma_0-iT_1}^{\sigma_0-iT} \right) \frac{(nx)^{-s}ds}{\chi^\lambda(s)} \ll \varepsilon_1. \quad (2.35)$$

By (2.25), (2.26) and (2.35),

$$\begin{aligned}
S_\lambda &= \sum_{n \leq N} \frac{d_\lambda(n)}{h_0^{\lambda-1}} \lim_{T \rightarrow \infty} \int_0^{h_0} \cdots \int_0^{h_0} dh_1 \cdots dh_{\lambda-1} \frac{1}{2\pi i} \int_{(a+h)^\lambda}^{(b+h)^\lambda} \frac{f(x)dx}{2\pi i} \int_{\sigma_0-iT}^{\sigma_0+iT} \frac{(nx)^{-s}ds}{\chi^\lambda(s)} \\
&\quad + S_\lambda + O(\varepsilon_1).
\end{aligned} \quad (2.36)$$

If $s = -1 + it$, $|t| > 1$, then by (2.22),

$$\left| \frac{1}{\chi(s)} \right| \ll |t|^{-\frac{3}{2}}. \quad (2.37)$$

Let c^\pm be the broken line from $\sigma_0 \pm iT$ to $-1 \pm iT$ and again to $-1 \pm i\infty$, it is easy to know that

$$\int_0^{h_0} \cdots \int_0^{h_0} dh_1 \cdots dh_{\lambda-1} \int_{(a+h)^\lambda}^{(b+h)^\lambda} f(x)dx \left(\int_{c^+} - \int_{c^-} \right) \frac{(nx)^{-s}ds}{\chi^\lambda(s)} \rightarrow 0 (T \rightarrow \infty) \quad (2.38)$$

uniformly for $n \leq N$. Therefore, by (2.36), (2.38),

$$S_\lambda = \sum_{n \leq N} \frac{d_\lambda(n)}{h_0^{\lambda-1}} \int_0^{h_0} \cdots \int_0^{h_0} dh_1 \cdots dh_{\lambda-1} \int_{(a+h)^\lambda}^{(b+h)^\lambda} f(x) J_{\lambda n}(x) dx + S_\lambda + O(\varepsilon_1) \quad (2.39)$$

holds for any $\varepsilon_1 > 0$ as $N \geq N(\varepsilon_1)$, where S_λ is (2.8) and

$$J_{\lambda n}(x) = \frac{1}{2\pi i} \int_c \frac{(nx)^{-s}ds}{\chi^\lambda(s)}, \quad (2.40)$$

the path c is an infinite broken line joining the points $-1 - i\infty, -1 - iT, \alpha_\lambda + 5/4 - iT, \alpha_\lambda + 5/4 + iT, -1 + iT, -1 + i\infty$.

It is known that (for example, (24), P.360 in [3])

$$\Gamma(s) = \sqrt{2\pi} e^{(s-1/2) \log s - s + v(s)}, \quad (2.41)$$

where

$$v(s) = \int_0^\infty \frac{(\{x\} - 1/2)dx}{x+s} = \int_0^\infty \frac{\varphi(x)dx}{(x+s)^2} \quad (2.42)$$

for $s \neq 0, -1, -2, \dots$,

$$\varphi(x) = - \int_0^x (\{y\} - 1/2)dy = \sum_{m=1}^\infty \frac{1 - \cos 2m\pi x}{2\pi^2 m^2}.$$

So that

$$\begin{aligned} v(s) &= \sum_{m=1}^\infty \frac{1}{2\pi^2 m^2} \int_0^\infty \frac{(1 - \cos 2m\pi x)dx}{(s+x)^2} \\ &= \frac{1}{s} \sum_{m=1}^\infty \frac{1}{2\pi^2 m^2} - \sum_{m=1}^\infty \frac{1}{2\pi^2 m^2} \int_0^\infty \frac{(\cos 2m\pi x)dx}{(s+x)^2} \\ &= \frac{c_1}{s} - \sum_{m=1}^\infty \frac{4}{(2\pi m)^3} \int_0^\infty \frac{(\sin 2m\pi x)dx}{(s+x)^3} \\ &= \frac{c_1}{s} + \frac{c_3}{s^3} + \dots + \frac{c_{2j_0+1}}{s^{2j_0+1}} + O\left(\frac{1}{s^{2j_0+2}}\right) \end{aligned} \quad (2.43)$$

for large $s \neq -m$, where

$$c_1 = \sum_{m=1}^\infty \frac{1}{2\pi^2 m^2} = \frac{\zeta(2)}{2\pi^2} = 1/12, \quad (2.44)$$

$$c_3 = - \sum_{m=1}^\infty \frac{4}{(2\pi m)^4} = -\frac{\zeta(4)}{4\pi^4} = -1/360, \quad (2.45)$$

$$c_5 = 1/1260, \quad (2.46)$$

$$c_7 = -1/1680, \quad (2.47)$$

...

By (2.41) and the first equality of (2.22),

$$\begin{aligned} \frac{1}{\chi^\lambda(\frac{s}{\lambda} + 1/2 - \frac{1}{2\lambda})} &= 2^\lambda (2\pi)^{-s - \frac{\lambda-1}{2}} \times \\ &\times \cos^\lambda\left(\frac{\pi}{2}\left(\frac{s}{\lambda} + 1/2 - \frac{1}{2\lambda}\right)\right) (\sqrt{2\pi})^\lambda e^{(s-1/2) \log(\frac{s}{\lambda} + 1/2 - \frac{1}{2\lambda}) - \lambda(\frac{s}{\lambda} + 1/2 - \frac{1}{2\lambda}) + \lambda v(\frac{s}{\lambda} + 1/2 - \frac{1}{2\lambda})} \\ &= (2\pi)^{-s} \left(2 \cos \frac{\pi}{2}\left(\frac{s}{\lambda} + 1/2 - \frac{1}{2\lambda}\right)\right)^\lambda \sqrt{2\pi} e^{(s-1/2) \log s - s + v(s) - (s-1/2) \log \lambda + \mu_\lambda(s)} \end{aligned}$$

$$= \sqrt{\lambda}(2\pi\lambda)^{-s}\Gamma(s)e^{\mu_\lambda(s)}(2\cos\frac{\pi}{2}(\frac{s}{\lambda} + 1/2 - \frac{1}{2\lambda}))^\lambda, \quad (2.48)$$

where

$$\begin{aligned} \mu_\lambda(s) &= (s - 1/2)\log(1 + \frac{\lambda - 1}{2s}) - \frac{\lambda - 1}{2} \\ &\quad + \lambda v(\frac{s}{\lambda} + 1/2 - \frac{1}{2\lambda}) - v(s) \\ &= \frac{1 - \lambda^2}{24s} + \frac{b_2^\lambda}{s^2} + \dots \end{aligned} \quad (2.49)$$

and

$$\begin{aligned} e^{\mu_\lambda(s)} &= 1 + \frac{1 - \lambda^2}{24s} + \frac{\beta_2^{(\lambda)}}{s^2} + \dots \\ &= \sum_{\alpha=0}^{\alpha_\lambda+1} \frac{\gamma_\alpha^{(\lambda)}}{(s-1)(s-2)\dots(s-\alpha)} + R_\lambda(s). \end{aligned} \quad (2.50)$$

The first term of the above sum is 1 and

$$R_\lambda(s) = O((\frac{1}{s - (\alpha_\lambda + 1)})^{\alpha_\lambda+2}). \quad (2.51)$$

Moreover,

$$(2\cos\frac{\pi}{2}(\frac{s}{\lambda} + 1/2 - \frac{1}{2\lambda}))^\lambda = 2\cos(\frac{\pi s}{2} + \frac{(\lambda - 1)\pi}{4}) + \rho_\lambda(s), \quad (2.52)$$

where

$$\rho_\lambda(s) = c_{\lambda-2}\cos\frac{\pi}{2}(\frac{(\lambda-2)s}{\lambda} + \theta_{\lambda-2}) + \dots + \rho_{1\lambda}(s), \quad (2.53)$$

$$\rho_{1\lambda}(s) = \begin{cases} c_{10\lambda}, & \lambda = 2\lambda_1 \\ c_{11\lambda}\cos(\frac{\pi s}{2\lambda} + \theta_{1\lambda}), & \lambda = 2\lambda_1 + 1 \end{cases}$$

Therefore, putting $s = \frac{s'}{\lambda} + 1/2 - \frac{1}{2\lambda}$ in the integral (2.40) and moving the integral line return to the original c , we have

$$\begin{aligned} J_{n\lambda}(x) &= 2\lambda^{-1/2}(nx)^{-\frac{1}{2} + \frac{1}{2\lambda}} \sum_{\alpha=0}^{\alpha_\lambda+1} \frac{\gamma_\alpha^{(\lambda)}}{2\pi i} \int_c \frac{(2\pi\lambda(nx)^{\frac{1}{\lambda}})^{-s}\Gamma(s)}{(s-1)(s-2)\dots(s-\alpha)} \times \\ &\quad \times \cos(\frac{\pi s}{2} + \frac{(\lambda-1)}{4})ds + J_{n\lambda\delta}(x), \end{aligned} \quad (2.54)$$

where

$$\begin{aligned}
J_{n\lambda\delta}(x) &= \frac{\lambda^{-1/2}(nx)^{-\frac{1}{2}+\frac{1}{2\lambda}}}{2\pi i} \int_c \Gamma(s)(2\pi\lambda(nx)^{\frac{1}{\lambda}})^{-s} \\
&\quad \times (e^{\mu_\lambda(s)}\rho_\lambda(s) + 2R_\lambda(s) \cos(\frac{\pi s}{2} + \frac{(\lambda-1)}{4}))ds \\
&= \frac{\lambda^{-1/2}(nx)^{-\frac{1}{2}+\frac{1}{2\lambda}}}{2\pi i} \int_{\alpha_\lambda+5/4-i\infty}^{\alpha_\lambda+5/4+i\infty} \\
&\ll (nx)^{-\frac{1}{2}+\frac{1}{2\lambda}-\frac{\alpha_\lambda+5/4}{\lambda}} \int_{-\infty}^{\infty} |\Gamma(\sigma_1+it)| \\
&\quad \times (|\rho_\lambda(\sigma_1+it)| + |R_\lambda(\sigma_1+it) \cos(\frac{\pi}{2}(\sigma_1+it) + \frac{\lambda-1}{4})|)dt, \\
\sigma_1 &= \alpha_\lambda + 5/4.
\end{aligned}$$

Since

$$\begin{aligned}
&|\Gamma(\sigma_1+it) \cos(\frac{\pi}{2}(\sigma_1+it)(1-\nu_0)+\theta)| \\
&\ll \begin{cases} |t|^{\sigma_1-1/2}, & \nu_0 = 0 \\ |t|^{\sigma_1-1/2}e^{-\frac{\pi|t|\nu_0}{2}}, & 0 < \nu_0 \leq 1 \end{cases} \\
&\ll \begin{cases} |t|^{\sigma_1-1/2}, & \nu_0 = 0 \\ e^{-|t|\nu_0}, & 0 < \nu_0 \leq 1 \end{cases}
\end{aligned}$$

for $|t| \geq 1$, then by (2.51) and (2.53),

$$\begin{aligned}
J_{n\lambda\delta}(x) &\ll (nx)^{-1/2+\frac{1}{2\lambda}-\frac{\alpha_\lambda+5/4}{\lambda}} \int_{-\infty}^{\infty} ((|t|+1)^{-5/4} + e^{-\frac{|t|}{\lambda}})dt \\
&\ll (nx)^{-1/2+\frac{1}{2\lambda}-\frac{\alpha_\lambda+5/4}{\lambda}}.
\end{aligned} \tag{2.55}$$

Finally, by (2.54) and (2.55), using $\frac{\Gamma(s)}{(s-1)(s-2)\dots(s-\alpha)} = \Gamma(s-\alpha)$, we obtain that

$$\begin{aligned}
J_{n\lambda}(x) &= 2\lambda^{-1/2}(nx)^{-1/2+\frac{1}{2\lambda}} \sum_{\alpha=0}^{\alpha_\lambda+1} \frac{\gamma_\alpha^{(\lambda)}(nx)^{\frac{-\alpha}{\lambda}}}{2\pi i(2\pi\lambda)^\alpha} \int_c (2\pi\lambda(nx)^{\frac{1}{\lambda}})^{-s} \Gamma(s) \\
&\quad \times \cos(\frac{\pi s}{2} + \frac{(\lambda-1)\pi}{4} + \frac{\pi\alpha}{2})ds + O((nx)^{-1/2+\frac{1}{2\lambda}-\frac{\alpha_\lambda+5/4}{\lambda}})
\end{aligned} \tag{2.56}$$

It is known that Mellin's formula:

$$\frac{1}{2\pi i} \int_c x^{-s} \Gamma(s) \cos(\frac{\pi s}{2} + \theta)ds = \cos(x + \theta), \tag{2.57}$$

hence

$$\begin{aligned}
J_{n\lambda}(x) &= 2\lambda^{-1/2}(nx)^{-1/2+\frac{1}{2\lambda}} \sum_{\alpha=0}^{\alpha_\lambda+1} \frac{\gamma_\alpha^{(\lambda)}(nx)^{\frac{-\alpha}{\lambda}}}{(2\pi\lambda)^\alpha} \\
&\quad \times \cos(2\pi\lambda(nx)^{\frac{1}{\lambda}} + \frac{(\lambda-1)\pi}{4} + \frac{\pi\alpha}{2}) + O((nx)^{-1/2+\frac{1}{2\lambda}-\frac{\alpha_\lambda+5/4}{\lambda}}) \\
&= 2\lambda^{-1/2}(nx)^{-1/2+\frac{1}{2\lambda}} \sum_{\alpha=0}^{\alpha_\lambda} \frac{\gamma_\alpha^{(\lambda)}(nx)^{\frac{-\alpha}{\lambda}}}{(2\pi\lambda)^\alpha} \\
&\quad \times \cos(2\pi\lambda(nx)^{\frac{1}{\lambda}} + \frac{(\lambda-1)\pi}{4} + \frac{\pi\alpha}{2}) + O((nx)^{-1/2+\frac{1}{2\lambda}-\frac{\alpha_\lambda+1}{\lambda}}). \quad (2.58)
\end{aligned}$$

By (2.39) and (2.58) we obtain (2.9). The proof of Lemma 2.2 is complete.

Lemma 2.3 is the case $\lambda = 2$ of Lemma 2.2. To evaluate concretely the first seventh coefficients $\gamma_0, \dots, \gamma_7$ in (2.13), it is sufficient to take $\lambda = 2$ in (2.49). We have

$$\begin{aligned}
\mu(s) &= \mu_2(s) = (s-1/2) \log(1 + \frac{1}{2s}) - 1/2 + 2v(\frac{s}{2} + 1/4) - v(s) \\
&= -\frac{1}{8s} - \frac{1}{16s^2} + \frac{1}{192s^3} + \frac{5}{128s^4} - \frac{1}{640s^5} - \frac{61}{768s^6} + \dots
\end{aligned}$$

and the calculation gives

$$e^{\mu_2(s)} = 1 + \sum_{\alpha=1}^{\alpha_0} \frac{\gamma_\alpha}{(s-1)(s-2)\dots(s-\alpha)} + R_2(s),$$

where $\gamma_1, \dots, \gamma_6$ are (2.13), $R_2(s)$ satisfies the corresponding (2.51). Proof of Lemma 2.4 is all the same.

It is known that (for example, (3.17) in [3])

$$\Delta(x) = \frac{x^{1/4}}{\pi\sqrt{2}} \sum_{n \leq N} d(n)n^{-3/4} \cos(4\pi\sqrt{nx} - \pi/4) + O(x^{1/2+\varepsilon}N^{-1/2}) + O(x^\varepsilon) \quad (2.59)$$

for any $\varepsilon > 0$. When x is not an integer, (see (15.24), [3]),

$$\begin{aligned}
\Delta(x) &= \frac{x^{1/4}}{\pi\sqrt{2}} \sum_{n=1}^{\infty} d(n)n^{-3/4} (\cos(4\pi\sqrt{nx} - \pi/4) \\
&\quad + \frac{3}{8(4\pi)}(nx)^{-1/2} \cos(4\pi\sqrt{nx} + \pi/4)) + O(x^{-3/4}). \quad (2.60)
\end{aligned}$$

Lemma 2.5 As $x \geq 0.9$, $0 < h_0 \ll x$, $N \geq N(\varepsilon_1)$,

$$\frac{1}{h_0} \int_0^{h_0} \Delta((x+h)^2)dh = \frac{1}{h_0} \sum_{n \leq N} d(n) \int_0^{h_0} \phi(n, (x+h)^2)dh + O(\varepsilon_1) \quad (2.61)$$

holds for any $\varepsilon_1 > 0$, where

$$\begin{aligned}\phi(n, x) &= \frac{x^{1/4}n^{-3/4}}{\sqrt{2\pi}} \sum_{\alpha=0}^{\alpha_0} \frac{\beta_\alpha \cos(4\pi\sqrt{nx} - \pi/4 + \alpha\pi/2)}{(4\pi\sqrt{nx})^\alpha} \\ &\quad + O(x^{1/4}n^{-3/4}(nx)^{-\frac{\alpha_0+1}{2}})\end{aligned}\tag{2.62}$$

with $\beta_0 = 1$, $\beta_1 = 3/8$, \dots , $\beta_{\alpha_0} = \dots$ for any integer number $\alpha_0 \geq 0$.

Proof: In fact, as $\alpha_0 = 0, 1$, (2.60) leads to (2.61); for the general $\alpha_0 \geq 0$, we have

$$\Delta((x+h)^2) = \frac{1}{2\pi i} \int_C \frac{(x+h)^{2s} \zeta^2(s)}{s} ds,$$

where the path C is from $1.1 - i\infty$ to $1.1 - i$ to $1/2 - i$ to $1/2 + i$ to $1.1 + i$ and to $1.1 + i\infty$. The proof is the same as Lemma 2.3. \square

A corollary of Lemma 2.5 is:

Lemma 2.6

$$\begin{aligned}\int_{h_1}^{h_2} f(h) \Delta((x+h)^2) dh \\ = \sum_{n \leq N} d(n) \int_{h_1}^{h_2} f(h) \phi(n, (x+h)^2) dh + O(\varepsilon_1)\end{aligned}\tag{2.63}$$

holds for any $\varepsilon_1 > 0$ as $N \geq N(\varepsilon_1)$, where $f(x) \in C^2[h_1, h_2]$, $x + h_1 \geq 2$, $h_1 < h_2 \ll x$, $\phi(n, x)$ is (2.62).

In the same way, like the proof of Lemma 2.2, for $0.9 < a \leq x \leq b < \infty$,

$$\begin{aligned}\frac{1}{h_0^{\lambda-1}} \int_0^{h_0} \cdots \int_0^{h_0} dh_1 \cdots dh_{\lambda-1} \Delta_\lambda((x+h)^\lambda) \\ = \frac{1}{h_0^{\lambda-1}} \sum_{n \leq N} d_\lambda(n) \int_0^{h_0} \cdots \int_0^{h_0} dh_1 \cdots dh_{\lambda-1} \phi_\lambda(n, (x+h)^\lambda)\end{aligned}$$

holds for any $\varepsilon_1 > 0$ as $N \geq N(\varepsilon_1)$, where

$$h = h_1 + \cdots + h_{\lambda-1}, \quad 0 < h_0 < a/2,$$

$$\begin{aligned}\phi_\lambda(n, x) &= \frac{x^{\frac{1}{2}-\frac{1}{2\lambda}}n^{-\frac{1}{2}-\frac{1}{2\lambda}}}{\sqrt{\lambda}\pi} \sum_{\alpha=0}^{\alpha_\lambda} \frac{\beta_\alpha^{(\lambda)} \cos(2\pi\lambda(nx)^{\frac{1}{\lambda}} + (\lambda-3)\pi/4 + \alpha\pi/2)}{(2\lambda\pi(nx)^{1/\lambda})^\alpha} \\ &\quad + O(x^{\frac{1}{2}-\frac{1}{2\lambda}}n^{-\frac{1}{2}-\frac{1}{2\lambda}}(nx)^{-\frac{\alpha_\lambda+1}{\lambda}}),\end{aligned}\tag{2.64}$$

$$\begin{aligned}\beta_0^{(\lambda)} &= 1, \\ \beta_1^{(\lambda)} &= \frac{\lambda-1}{2} + \frac{1-\lambda^2}{24}, \\ &\dots\end{aligned}$$

for any positive integer number α_λ .

3 SOME LEMMAS (II)

Lemma 3.1 (see(1.17), [2]). For $n \geq 3$,

$$d(n) \leq \exp(O(\frac{\log n}{\log \log n})) = n^{\varepsilon(n)}, \quad \varepsilon(n) = O(\frac{1}{\log \log n}) \quad (3.1)$$

The following Lemma 3.2 is a generalization of Theorem B.1 in [3], also the proof belong to it.

Lemma 3.2 Let

$$\psi(V, b, m_0, x) = \frac{1}{\sqrt{V}} \sum_{j=-\infty}^{\infty} e^{-\frac{\pi(j+b)^2}{V}} e(jx)(j+b)^{m_0} \quad (3.2)$$

where $\operatorname{Re} V > 0$, b is real, $x = x_1 + ix_2$, m_0 is a negative integer, then

$$\psi(V, b, m_0, x) = \sum_{j=-\infty}^{\infty} e^{-\pi V(j+x)^2} e(-b(x+j)) p_{m_0}(x, j) \quad (3.3)$$

where

$$\begin{aligned} p_{m_0}(x, j) &= \int_{-\infty}^{\infty} e^{-\pi \theta^2} (Vi(x+j) + \sqrt{V}\theta)^{m_0} d\theta \\ &= \sum_{\nu=0}^{\lfloor \frac{m_0}{2} \rfloor} \binom{m_0}{2\nu} \frac{(2\nu-1)!! V^\nu}{(2\pi)^\nu} (Vi(x+j))^{m_0-2\nu}, \\ (-1)!! &= 1. \end{aligned} \quad (3.4)$$

Proof: Let

$$I_N = \frac{1}{\sqrt{V}} \int_{C_N} \frac{e^{-\frac{\pi(z+b)^2}{V}} e(zx)(z+b)^{m_0} dz}{e(z) - 1}$$

where C_N is the rectangle with vertices at $N + 1/2 \pm i$, $-(N + 1/2) \pm i$, N is a positive integer, then

$$I_N = \frac{1}{\sqrt{V}} \sum_{j=-N}^N e^{-\frac{\pi(j+b)^2}{V}} e(jx)(j+b)^{m_0} \rightarrow \psi(V, b, m_0, x) \quad (N \rightarrow \infty)$$

On the other hand, since

$$\int_{\pm(N+1/2)-i}^{\pm(N+1/2)+i} \frac{e^{-\pi(z+b)^2} e(zx)(z+b)^{m_0} dz}{e(z) - 1} \ll \int_{-1}^1 \frac{e^{-\varepsilon_0 N} dy}{1 + e^{-2\pi y}} \rightarrow 0 \quad (N \rightarrow \infty), \quad \varepsilon_0 > 0,$$

then

$$\begin{aligned}
\psi &= \frac{1}{\sqrt{V}} \left(\int_{-\infty-i}^{\infty-i} + \int_{-\infty+i}^{\infty+i} \right) \frac{e^{-\frac{\pi(z+b)^2}{V}} e(zx)(z+b)^{m_0} dz}{e(z) - 1} \\
&= \frac{1}{\sqrt{V}} \sum_{j=-\infty}^{-1} \int_{-\infty-i}^{\infty-i} e^{-\frac{\pi(z+b)^2}{V}} e(z(x+j))(z+b)^{m_0} dz \\
&\quad + \frac{1}{\sqrt{V}} \sum_{j=0}^{\infty} \int_{-\infty+i}^{\infty+i} e^{-\frac{\pi(z+b)^2}{V}} e(z(x+j))(z+b)^{m_0} dz.
\end{aligned}$$

Putting $z = -b + z'$, and moving the integral line to $(-\infty, \infty)$, we have

$$\psi = \sum_{j=-\infty}^{\infty} e(-b(x+j)) I_j,$$

where

$$\begin{aligned}
I_j &= \frac{1}{\sqrt{V}} \int_{-\infty}^{\infty} e^{-\frac{\pi z^2}{V}} e(z(x+j)) z^{m_0} dz \\
&= \frac{1}{\sqrt{V}} e^{-\pi v(x+j)^2} \int_{-\infty}^{\infty} e^{-\frac{\pi(z-iV(x+j))^2}{V}} z^{m_0} dz \\
&= e^{-\pi V(x+j)^2} p_{m_0}(x, j),
\end{aligned}$$

$p_{m_0}(x, j)$ is (3.4). The proof is complete. \square

Lemma 3.3 *Let $f(x) \in C^2(a - \varepsilon, b]$, $\varepsilon > 0$*

$$\sigma = \sum_{a \leq n \leq b} f(n), \quad (3.5)$$

$$M_\nu = \max_{a \leq n \leq b} |f^{(\nu)}(x)|, \quad \nu = 1, 2, \quad (3.6)$$

then

$$\sigma = \sum_{-N \leq n \leq N} \int_a^b f(x) e(nx) dx + \sigma_0(a) + \sigma_0(b) + O\left(\frac{M_1 + (b-a)M_2}{N}\right) \quad (3.7)$$

for any natural N , where $\sigma_0(X)$, $X = a$ or b , such that the following condition: If X is an integer, then

$$\sigma_0(X) = \frac{f(X)}{2}; \quad (3.8)$$

if X is not an integer, then

$$\sigma_0(X) \ll \frac{1/\{X\} + 1/(1 - \{X\})}{N}. \quad (3.9)$$

Proof: We have

$$\sigma = \lim_{\varepsilon \rightarrow 0_+} \sum_{a-\varepsilon \leq n \leq b} f(n) = \lim_{\varepsilon \rightarrow 0_+} \int_{a-\varepsilon}^b f(x) d(x + \frac{1}{2} - \{x\}) = \int_a^b f(x) dx + I_1, \quad (3.10)$$

where

$$I_1 = \lim_{\varepsilon \rightarrow 0_+} \int_{a-\varepsilon}^b f(x) d(\frac{1}{2} - \{x\}) = \lim_{\varepsilon \rightarrow 0_+} f(x) (\frac{1}{2} - \{x\}) \Big|_{a-\varepsilon}^b + I_2, \quad (3.11)$$

with

$$\begin{aligned} I_2 &= - \int_a^b f'(x) (\frac{1}{2} - \{x\}) dx \\ &= - \int_a^b f'(x) (\int_0^x (\frac{1}{2} - \{y\}) dy)' dx \\ &= -f'(x) \int_0^x (\frac{1}{2} - \{y\}) dy \Big|_a^b + \int_a^b f''(x) (\int_0^x (\frac{1}{2} - \{y\}) dy) dx. \end{aligned}$$

Using

$$\int_0^x (\frac{1}{2} - \{y\}) dy = \sum_{n=1}^{\infty} \frac{1 - \cos 2n\pi x}{2\pi^2 n^2}, \quad (3.12)$$

we have

$$\begin{aligned} I_2 &= -f'(x) \sum_{n=1}^N \frac{1 - \cos 2n\pi x}{2\pi^2 n^2} \Big|_a^b + \int_a^b f''(x) \sum_{n=1}^N \frac{1 - \cos 2n\pi x}{2\pi^2 n^2} dx + O(\frac{M_1 + (b-a)M_2}{N}) \\ &= 2 \sum_{n=1}^N \int_a^b f(x) \cos 2n\pi x dx - \sum_{n=1}^N \frac{f(x) \sin 2n\pi x}{\pi n} \Big|_a^b + O(\frac{M_1 + (b-a)M_2}{N}). \end{aligned} \quad (3.13)$$

Since $2 \cos 2\pi x = e(x) + e(-x)$, then (3.7) is obtained by (3.10), (3.11) and (3.13), where

$$\sigma_0(a) = -f(a) (\lim_{\varepsilon \rightarrow 0_+} (\frac{1}{2} - \{a - \varepsilon\}) - \sum_{n=1}^N \frac{\sin 2n\pi a}{\pi n}), \quad (3.14)$$

$$\sigma_0(b) = f(b) (\frac{1}{2} - \{b\} - \sum_{n=1}^N \frac{\sin 2n\pi b}{\pi n}). \quad (3.15)$$

Let $X = a$ or b , clearly, if X is an integer, then $\sigma_0(X) = \frac{1}{2}f(X)$. If $\{X\} = \frac{1}{2}$, then

$\sigma_0(X)$ is vanishing. Suppose $0 < \{X\} < \frac{1}{2}$. We have

$$\begin{aligned}
\frac{1}{2} - \{X\} - \sum_{n=1}^N \frac{\sin 2n\pi X}{\pi n} &= \sum_{n=N+1}^{\infty} \frac{\sin 2n\pi \{X\}}{\pi n} \\
&= \lim_{N_1 \rightarrow \infty} \frac{1}{\pi} \int_{N+1}^{N_1} \frac{1}{u} d \sum_{n \leq u} \sin 2n\pi \{X\} \\
&\ll \frac{1}{N} \left(\sum_{n \leq N+1} \sin 2n\pi \{X\} + \int_{N+1}^{\infty} \frac{1}{u^2} \left(\sum_{n \leq u} \sin 2n\pi \{X\} \right) du \right) \\
&\ll \frac{1}{N} \left(\frac{1}{\{X\}} + \frac{1}{1 - \{X\}} \right).
\end{aligned}$$

The case $\frac{1}{2} < \{X\} < 1$ is the same. The proof is complete. \square

Lemma 3.4 *Let*

$$H = H(X, \alpha, \beta, a, b) = \sum_{a < n \leq b} d(n) n^{-\alpha} \cos(4\pi \sqrt{nX} + \beta), \quad (3.16)$$

where $0.9 < a < b \leq X(\log X)^C$, $C \geq 0$; α, β are real, X is large positive number, then

$$H \ll (a^{1/4-\alpha} + b^{1/4-\alpha}) X^{1/4+\varepsilon}, \quad (3.17)$$

where

$$\varepsilon = \varepsilon(X) = O\left(\frac{1}{\log \log X}\right). \quad (3.18)$$

Proof: Let $h_0 = \frac{1}{4}$ and

$$H' = \frac{1}{h_0} \int_0^{h_0} dh \sum_{\sqrt{a+h} < \sqrt{n} \leq \sqrt{b+h}} d(n) n^{-\alpha} \cos(4\pi \sqrt{nX} + \beta), \quad (3.19)$$

then from Lemma 3.1

$$\begin{aligned}
H - H' &= \frac{1}{h_0} \int_0^{h_0} dh \left(\sum_{\sqrt{a} < \sqrt{n} \leq \sqrt{b}} - \sum_{\sqrt{a+h} < \sqrt{n} \leq \sqrt{b+h}} \right) d(n) n^{-\alpha} \cos(4\pi \sqrt{nX} + \beta) \\
&\ll \frac{1}{h_0} \int_0^{h_0} \left(\sum_{|\sqrt{n} - \sqrt{a}| \leq h} + \sum_{|\sqrt{n} - \sqrt{b}| \leq h} \right) d(n) n^{-\alpha} \\
&\ll a^{\frac{1}{2}-\alpha+\varepsilon} + b^{\frac{1}{2}-\alpha+\varepsilon},
\end{aligned}$$

where $\varepsilon = \varepsilon(X)$ is (3.18). So that

$$H = \frac{1}{h_0} \int_0^{h_0} dh \sum_{\sqrt{a+h} < \sqrt{n} \leq \sqrt{b+h}} d(n) n^{-\alpha} \cos(4\pi \sqrt{nX} + \beta) + O(a^{\frac{1}{2}-\alpha+\varepsilon} + b^{\frac{1}{2}-\alpha+\varepsilon}) \quad (3.20)$$

Taking $f(n) = n^{-\alpha} \cos(4\pi\sqrt{nX} + \beta)$ in Lemma 2.3, we have

$$H = H_l + H_{\Delta N} + O(\varepsilon_1) + O(a^{\frac{1}{2}-\alpha+\varepsilon} + b^{\frac{1}{2}-\alpha+\varepsilon}) \quad (3.21)$$

holds for any $\varepsilon_1 > 0$ as N large, where

$$\begin{aligned} H_l &= \frac{1}{h_0} \int_0^{h_0} dh \int_{(\sqrt{a}+h)^2}^{(\sqrt{b}+h)^2} u^{-\alpha} (\log u + 2\gamma) \cos(4\pi\sqrt{uX} + \beta) du \\ &= \frac{2}{h_0} \int_0^{h_0} dh \int_{\sqrt{a}+h}^{\sqrt{b}+h} u^{1-2\alpha} (\log u^2 + 2\gamma) \cos(4\pi\sqrt{X}u + \beta) du \\ &\ll (a^{\frac{1}{2}-\alpha+\varepsilon} + b^{\frac{1}{2}-\alpha+\varepsilon})/\sqrt{X}, \end{aligned} \quad (3.22)$$

$$\begin{aligned} H_{\Delta N} &= \sum_{n \leq N} \frac{d(n)}{h_0} \int_0^{h_0} dh \int_{(\sqrt{a}+h)^2}^{(\sqrt{b}+h)^2} u^{-\alpha} \cos(4\pi\sqrt{uX} + \beta) \Theta(nu) du \\ &= 2\sqrt{2} \sum_{n \leq N} d(n) n^{-\frac{1}{4}} \sum_{\nu=0}^1 \frac{\gamma_\nu}{(4\pi\sqrt{n})^\nu} j_{n\nu} + \delta_{\Delta N}, \end{aligned} \quad (3.23)$$

$$\delta_{\Delta N} \ll \frac{1}{h_0} \sum_{n \leq N} d(n) n^{-\frac{1}{4}} \int_0^{h_0} dh \int_{(\sqrt{a}+h)^2}^{(\sqrt{b}+h)^2} \frac{u^{-\alpha-1}}{n} du \ll a^{-\alpha} + b^{-\alpha}, \quad (3.24)$$

$$\begin{aligned} j_{n\nu} &= \frac{1}{h_0} \int_0^{h_0} dh \int_{\sqrt{a}+h}^{\sqrt{b}+h} u^{\frac{1}{2}-2\alpha-\nu} \cos 4\pi(\sqrt{X}u + \beta) \cos(4\pi\sqrt{n}u + \frac{\pi}{4} + \frac{\pi\nu}{2}) du \\ &= j_{n\nu}^+ + j_{n\nu}^-, \end{aligned} \quad (3.25)$$

$$\begin{aligned} j_{n\nu}^\pm &= \frac{1}{2h_0} \int_0^{h_0} dh \int_{\sqrt{a}+h}^{\sqrt{b}+h} u^{\frac{1}{2}-2\alpha-\nu} \cos(4\pi(\sqrt{n} \pm \sqrt{X})u + \theta^\pm) du \\ &= \frac{1}{2h_0} \int_0^{h_0} \frac{(u+h)^{\frac{1}{2}-2\alpha-\nu} \sin(4\pi(\sqrt{n} \pm \sqrt{X})(u+h) + \theta^\pm)}{4\pi(\sqrt{n} \pm \sqrt{X})} \Big|_{\sqrt{a}}^{\sqrt{b}} \\ &\quad - \frac{c}{2h_0} \int_0^{h_0} \frac{dh}{\sqrt{n} \pm \sqrt{X}} \int_{\sqrt{a}}^{\sqrt{b}} (u+h)^{-\frac{1}{2}-2\alpha-\nu} \sin(4\pi(\sqrt{n} \pm \sqrt{X})(u+h) + \theta^\pm) du \end{aligned} \quad (3.26)$$

for $\sqrt{n} \neq \sqrt{X}$. If $n > 2X$, then integrating by parts for h , we have

$$j_{n\nu}^\pm \ll \frac{a^{\frac{1}{4}-\alpha} + b^{\frac{1}{4}-\alpha}}{(\sqrt{n} \pm \sqrt{X})^2} \ll \frac{a^{\frac{1}{4}-\alpha} + b^{\frac{1}{4}-\alpha}}{n}. \quad (3.27)$$

Therefore, it follows from (3.23), (3.24) and (3.27) that

$$H_{\Delta N} = 2\sqrt{2} \sum_{n \leq 2X} d(n) n^{-\frac{1}{4}} \sum_{\nu=0}^1 \frac{\gamma_\nu}{(4\pi\sqrt{n})^\nu} (j_{n\nu}^+ + j_{n\nu}^-) + O(a^{\frac{1}{4}-\alpha} + b^{\frac{1}{4}-\alpha}). \quad (3.28)$$

If $|n - X| \leq 2$, then by (3.25),

$$j_{nv}^{\pm} \ll \int_{\sqrt{a}}^{\sqrt{b}+1} u^{\frac{1}{2}-2\alpha} du \ll a^{\frac{3}{4}-\alpha} + b^{\frac{3}{4}-\alpha}. \quad (3.29)$$

If $|n - X| > 2$, then by (3.26),

$$j_{nv}^{\pm} \ll \frac{1}{|\sqrt{n} - \sqrt{X}|} (a^{\frac{1}{4}-\alpha} + b^{\frac{1}{4}-\alpha}). \quad (3.30)$$

Hence, by (3.27), (3.28), (3.29) and Lemma 3.1 we obtain

$$\begin{aligned} H_{\Delta N} &\ll \sum_{n \leq 2X, |n-X| > 2} \frac{d(n)n^{-\frac{1}{4}}}{|\sqrt{n} - \sqrt{X}|} (a^{1/4-\alpha} + b^{1/4-\alpha}) + \sum_{|n-X| \leq 2} d(n)n^{-\frac{1}{4}} (a^{3/4-\alpha} + b^{3/4-\alpha}) \\ &\ll \sum_{n \leq \frac{1}{2}X} d(n)n^{-\frac{1}{4}} \frac{a^{1/4-\alpha} + b^{1/4-\alpha}}{\sqrt{X}} \\ &\quad + (a^{1/4-\alpha} + b^{1/4-\alpha}) X^{\frac{1}{4}+\varepsilon(X)} \sum_{X/2 < n \leq 2X, |n-X| > 2} \frac{1}{|n-X|} + (a^{3/4-\alpha} + b^{3/4-\alpha}) X^{-\frac{1}{4}+\varepsilon(X)} \\ &\ll (a^{1/4-\alpha} + b^{1/4-\alpha}) X^{\frac{1}{4}+\varepsilon(X)} \end{aligned} \quad (3.31)$$

for $a < b \ll (\log X)^C$. (3.20), (3.21), (3.22) and (3.31) implicate (3.17). The proof is complete. \square

Lemma 3.5 Suppose a, b, α, β, X be as Lemma 3.4,

$$H = \sum_{a < n \leq b} d(n)n^{-\alpha} f(n) \cos(4\pi\sqrt{nX} + \beta), \quad f(u) \ll M,$$

$$f'(u) \ll \frac{M}{\sqrt{Xu}}, \quad u \in [a, b+1], \quad (3.32)$$

then

$$H \ll M(a^{1/4-\alpha} + b^{1/4-\alpha}) X^{\frac{1}{4}+\varepsilon(X)}. \quad (3.33)$$

Proof: In fact, by Lemma 3.4,

$$\begin{aligned} H &= \int_a^b f(u) d \sum_{a < n \leq u} d(n)n^{-\alpha} f(n) \cos(4\pi\sqrt{nX} + \beta) \\ &\ll |f(b)| \sum_{a < n \leq b} d(n)n^{-\alpha} f(n) \cos(4\pi\sqrt{nX} + \beta) \\ &\quad + \int_a^b |f'(u)| \sum_{a < n \leq u} d(n)n^{-\alpha} f(n) \cos(4\pi\sqrt{nX} + \beta) du \\ &\ll M(a^{1/4-\alpha} + b^{1/4-\alpha}) X^{\frac{1}{4}+\varepsilon(X)} (1 + \int_a^b \frac{du}{\sqrt{Xu}}) \\ &\ll M(a^{1/4-\alpha} + b^{1/4-\alpha}) X^{\frac{1}{4}+\varepsilon(X)}. \end{aligned}$$

The proof is complete. \square

Let

$$\eta = \eta_1 + \cdots + \eta_K,$$

define

$$f(\eta) \Big|_{\eta_k} = f(\eta_1 + \cdots + \eta_K) \Big|_{\eta_1=0}^{\eta_1=\nu_0} \cdots \Big|_{\eta_K=0}^{\eta_K=\nu_0} \quad (3.34)$$

Also we define

$$f(\eta_1) \Big|_{\eta_1=0}^{\eta_1=\nu_0} = |f(\nu_0)| + |f(0)|$$

generally,

$$f(\eta) \Big|_{\eta_K} = \sum_{\eta_1=0, \nu_0} \cdots \sum_{\eta_K=0, \nu_0} |f(\eta)| \quad (3.35)$$

It is obvious that if $f(\eta) \ll M$, then

$$f(\eta) \Big|_{\eta_K} \ll f(\eta) \Big|_{\eta_K} \ll M2^K; \quad (3.36)$$

if $f^{(k)}(\eta) \ll M$, $0 \leq k \leq K$, then

$$\begin{aligned} f(\eta) \Big|_{\eta_K} &= \left(\int_0^{\nu_0} \cdots \int_0^{\nu_0} f^{(k)}(\eta) d\eta_1 \cdots d\eta_K \right) \Big|_{\eta_{k+1}=0}^{\eta_{k+1}=\nu_0} \cdots \Big|_{\eta_K=0}^{\eta_K=\nu_0} \\ &\ll M2^{K-k} \int_0^{\nu_0} \cdots \int_0^{\nu_0} d\eta_1 \cdots d\eta_K = M2^{K-k} \nu_0^k. \end{aligned} \quad (3.37)$$

Lemma 3.6 Suppose $f(z)$ is analytic for $|z| \leq \frac{1}{\sqrt{L}}$, and $|f(z)| \leq M$, then

$$f(\eta) \Big|_{\eta_k} = f(\eta_1 + \cdots + \eta_K) \Big|_{\eta_1=0}^{\eta_1=\nu_0} \cdots \Big|_{\eta_K=0}^{\eta_K=\nu_0} \ll MU^{-99} \quad (3.38)$$

for large U , $L \asymp \log U$, $0 < \nu_0 \leq L^{-2}$, $K = \lceil \frac{C_0 L}{\log L} \rceil$, $C_0 \geq 200$.

Proof: Clearly,

$$0 \leq \eta_1 + \cdots + \eta_K \leq \nu_0 K \ll \frac{1}{L \log L}. \quad (3.39)$$

Moreover,

$$\begin{aligned} f(\eta) \Big|_{\eta_K} &= \int_0^{\nu_0} \cdots \int_0^{\nu_0} f^{(k)}(\eta) d\eta_1 \cdots d\eta_K \\ &= \int_0^{\nu_0} \cdots \int_0^{\nu_0} d\eta_1 \cdots d\eta_K \frac{K!}{2\pi i} \int_{|z|=\frac{1}{\sqrt{L}}} \frac{f(z) dz}{(z - \eta)^{K+1}} \\ &\ll K! \nu_0^K \int_{|z|=\frac{1}{\sqrt{L}}} \frac{M |dz|}{\left(\frac{1}{\sqrt{L}} - \frac{1}{L \log L} \right)^{K+1}} \\ &\ll K! L^{-2K} M (2\sqrt{L})^K \leq M (2KL^{-3/2})^K \\ &= M e^{-K \log \frac{L^{3/2}}{2K}} \\ &\ll M \exp\left(-\frac{200L}{\log L} (\log L^{1/2} + O(\log \log L))\right) \\ &\ll MU^{-99}, \end{aligned}$$

for $L \asymp \log U$. The proof is complete. \square

4 SOME LEMMAS (III)

Lemma 4.1 *Let*

$$G_\lambda(\sqrt{n}, \eta) = e^{-\frac{4\pi(\sqrt{n} + \lambda(\sqrt{U} + \eta + \nu))^2}{A^2}}, \quad (4.1)$$

$$W_\lambda(\eta) = e(-\lambda(\sqrt{U} + \eta + \nu)^2), \quad (4.2)$$

$$S_\lambda(\eta) = W_\lambda(\eta) \sum_{\sqrt{n} \leq \sqrt{U}} d(n) n^{-\frac{1}{4}} G_\lambda(\sqrt{n}, \eta) \sum_{\alpha=0}^2 \frac{\gamma_\alpha(-i)^\alpha e(-2\sqrt{n}(\sqrt{U} + \eta + \nu) - 1/8)}{(4\pi\sqrt{n}(\sqrt{n} + \sqrt{U} + \eta + \nu))^\alpha}, \quad (4.3)$$

$$S_\lambda^*(\eta) = W_\lambda(\eta) \sum_{\sqrt{U} < \sqrt{n} \leq (\lambda_0 - \lambda)\sqrt{U}} d(n) n^{-\frac{1}{4}} G_\lambda(\sqrt{n}, \eta) e(-2\sqrt{n}(\sqrt{U} + \eta + \nu) - 1/8), \quad (4.4)$$

where $\nu = O(L^{-1})$ is real, A is (1.10), η is (1.15),

$$1 \leq \lambda_0 \ll L, \quad 0 \leq \lambda \leq \lambda_0 - 1. \quad (4.5)$$

Then

$$S_0^*(\eta) = \sum_{\lambda=1}^{\lambda_0-1} S_\lambda(\eta) + O(U^{-3/4+\varepsilon}), \quad \varepsilon = O\left(\frac{1}{\log L}\right). \quad (4.6)$$

(note that O_η is (1.28).)

Proof: Let

$$U = [X], \quad 0 < h \leq h_0 = \frac{1}{\sqrt{U}L^2}, \quad (4.7)$$

then

$$0 < (\sqrt{U} + h)^2 - U < L^{-3/2}, \quad (4.8)$$

$$\{\sqrt{U}^2\} = \{U\} = 0,$$

Hence there doesn't exist any integer in the interval $(U, (\sqrt{U} + h)^2]$, that is

$$\sum_{\sqrt{U} < \sqrt{n} \leq \sqrt{U} + h} = 0.$$

In the same way,

$$\sum_{(\lambda_0 - \lambda)\sqrt{U} < \sqrt{n} \leq (\lambda_0 - \lambda)\sqrt{U} + h} = 0.$$

Noticing $e(n) = 1$, we have

$$S_\lambda^*(\eta) = \frac{e(-1/8)W_\lambda(\eta)}{h_0}$$

$$\times \int_0^{h_0} dh \sum_{\sqrt{U}+h < \sqrt{n} \leq (\lambda_0 - \lambda)\sqrt{U}+h} d(n)n^{-1/4} G_\lambda(\sqrt{n}, \eta) e(n - 2\sqrt{n}(\sqrt{U} + \eta + \nu)). \quad (4.9)$$

By Lemma 2.3

$$\begin{aligned} S_\lambda^*(\eta) &= \sum_{n \leq N} \frac{e(-1/8)W_\lambda(\eta)d(n)}{h_0} \int_0^{h_0} dh \int_{(\sqrt{U}+h)^2}^{((\lambda_0 - \lambda)\sqrt{U}+h)^2} \\ &\quad x^{-1/4} G_\lambda(\sqrt{n}, \eta) \Theta(nx) e(x - 2\sqrt{x}(\sqrt{U} + \eta + \nu)) dx \\ &\quad + \delta_\lambda(\eta) + O(\varepsilon_1), \end{aligned} \quad (4.10)$$

holds for any $\varepsilon_1 > 0$, as $N \geq N(\varepsilon_1)$, where $\Theta(nx)$ is (2.12),

$$\begin{aligned} \delta_\lambda(\eta) &= \frac{e(-1/8)W_\lambda(\eta)}{h_0} \int_0^{h_0} dh \int_{(\sqrt{U}+h)^2}^{((\lambda_0 - \lambda)\sqrt{U}+h)^2} x^{-1/4} G_\lambda(\sqrt{n}, \eta) \\ &\quad \times e(x - 2\sqrt{x}(\sqrt{U} + \eta + \nu)) (\log x + 2\gamma) dx \\ &= \frac{2e(-1/8)W_\lambda(\eta)}{h_0} \int_0^{h_0} dh \int_{\sqrt{U}+h}^{(\lambda_0 - \lambda)\sqrt{U}+h} x^{1/2} G_\lambda(\sqrt{n}, \eta) \\ &\quad \times e(x^2 - 2\sqrt{x}(\sqrt{U} + \eta + \nu)) (\log x^2 + 2\gamma) dx. \end{aligned} \quad (4.11)$$

Moving the integral line of the inner integral we obtain

$$\begin{aligned} \delta_\lambda(\eta) &= \frac{2e(-1/8)W_\lambda(\eta)}{h_0} \int_0^{h_0} dh \left(\int_{\sqrt{U}+h}^{\sqrt{U}+h+z_0} + \int_{\sqrt{U}+h+z_0}^{(\lambda_0 - \lambda)\sqrt{U}+h+z_0} + \int_{(\lambda_0 - \lambda)\sqrt{U}+h+z_0}^{(\lambda_0 - \lambda)\sqrt{U}+h} \right) \\ &\quad \times z^{\frac{1}{2}} G_\lambda(\sqrt{n}, \eta) (\log z^2 + 2\gamma) e(z^2 - 2z(\sqrt{U} + \eta + \nu)) dz \\ &= \frac{2e(-1/8)W_\lambda(\eta)}{h_0} \int_0^{h_0} dh (I_{1h}(\eta) + I_{2h}(\eta) + I_{3h}(\eta)), \quad \text{say} \end{aligned} \quad (4.12)$$

where

$$z_0 = Le\left(\frac{1}{8}\right). \quad (4.13)$$

Since

$$\begin{aligned} G_\lambda(z, \eta) &= e^{-\frac{4\pi(z + \lambda(\sqrt{U} + \eta + \nu))^2}{A^2}} \\ &= \int_{-\infty}^{\infty} e^{-\pi\theta^2} e\left(\frac{2\theta(z + \lambda(\sqrt{U} + \eta + \nu))}{A}\right) d\theta \\ &= \int_{-L}^L e^{-\pi\theta^2} e\left(\frac{2\theta\lambda(\sqrt{U} + \nu)}{A}\right) e\left(\frac{2(z + \lambda\eta)\theta}{A}\right) d\theta + O(e^{-L^2}), \end{aligned} \quad (4.14)$$

then by (4.12),

$$W_\lambda(\eta) I_{1h}(\eta) = \int_{-L}^L e^{-\pi\theta^2} e\left(\frac{2\theta\lambda(\sqrt{U} + \nu)}{A}\right) J_{1\theta}(\eta) + O(e^{-L^2}), \quad (4.15)$$

where

$$J_{1\theta}(\eta) = W_\lambda(\eta) \int_{\sqrt{U}+h}^{\sqrt{U}+h+z_0} f_\theta(z) e(z^2 - 2z(\sqrt{U} + \eta + \nu) + \frac{2(z + \lambda\eta)\theta}{A}) dz$$

$$= \int_0^{z_0} f_\theta(\sqrt{U} + h + z) e((\sqrt{U} + h + z)^2 + 2(\sqrt{U} + h + z)(-\sqrt{U} - \nu + \frac{\theta}{A})) \omega(z, \eta) dz, \quad (4.16)$$

$$f_\theta(z) = z^{1/2}(\log z^2 + 2\gamma), \quad (4.17)$$

$$\begin{aligned} \omega(z, \eta) &= e(2\eta(-\sqrt{U} - h - z + \frac{\lambda\theta}{A})) W_\lambda(\eta) \\ &= e(2\eta(-\sqrt{U} - h - z + \frac{\lambda\theta}{A}) - \lambda(\sqrt{U} + \eta + \nu)^2) \\ &= e(-\lambda(\sqrt{U} + \nu)^2) g_{z\theta}(\eta), \end{aligned} \quad (4.18)$$

$$\begin{aligned} g_{z\theta}(\eta) &= e(-2(\lambda + 1)\eta\sqrt{U} + 2\eta(-h - z - \nu\lambda + \frac{\lambda\theta}{A}) - \lambda\eta^2) \\ &= e(2\eta(-h - z + \frac{\lambda\theta}{A}) - \lambda\eta^2). \end{aligned} \quad (4.19)$$

The last equality is given by

$$e(-2(\lambda + 1)\eta\sqrt{U}) = e(\eta\sqrt{U}) = 1 \quad (4.20)$$

where $\eta = \eta_1 + \dots + \eta_K$, $\eta_j = 0$ or k_1/\sqrt{U} , $j = 1, 2, \dots, K$. Thus,

$$g_{z\theta}(w) \ll e^{4\pi|w|(|\frac{\lambda\theta}{A}| + |h| + |z| + \lambda|w|^2)} \ll e^{8\pi\sqrt{L}} \quad (4.21)$$

for $|w| \leq \frac{1}{\sqrt{L}}$, $|z| \leq L$, $\lambda \ll L$, $|\theta| \ll L$, $h = (\frac{1}{\sqrt{UL}})$, $A = C_0\sqrt{UL}$. By Lemma 3.6, we have

$$g_{z\theta}(\eta)|_{\eta_K} \ll e^{8\pi\sqrt{L}} U^{-99} \ll U^{-98} \quad (4.22)$$

for $\nu_0 = k_1/\sqrt{U} \ll L^{-2}$. It follows from (4.15), (4.16), (4.17), (4.18), (4.19) and (4.22) that

$$W_\lambda(\eta) I_{1h}(\eta)|_\eta \ll U^{-90}. \quad (4.23)$$

In the same way,

$$W_\lambda(\eta) I_{3h}(\eta)|_\eta \ll U^{-90}. \quad (4.24)$$

By (4.13),

$$\begin{aligned} W_\lambda(\eta) I_{2h}(\eta) &= W_\lambda(\eta) \int_{\sqrt{U}+h+z_0}^{(\lambda_0-\lambda)\sqrt{U}+h+z_0} f_\theta(z) G_\lambda(\sqrt{n}, \eta) e(z^2 - 2z(\sqrt{U} + \eta + \nu)) dz \\ &\ll \int_{\sqrt{U}+h}^{(\lambda_0-\lambda)\sqrt{U}+h} \sqrt{UL} |e((x + z_0)^2 - 2(z_0 + x)(\sqrt{U} + \eta + \nu))| dx \\ &= \sqrt{UL} \int_{\sqrt{U}+h}^{(\lambda_0-\lambda)\sqrt{U}+h} |e(z_0^2 + 2z_0(x - (\sqrt{U} + \eta + \nu)))| dx \\ &= \sqrt{UL} \int_{\sqrt{U}+h}^{(\lambda_0-\lambda)\sqrt{U}+h} e^{-2\pi L^2 - 2\sqrt{2}\pi(x - (\sqrt{U} + \eta + \nu))} dx \end{aligned}$$

$$\ll \sqrt{UL}e^{-L^2} = O_\eta(U^{-90}) \quad (4.25)$$

for $z_0 = Le(\frac{1}{z})$. It follows from (4.12), (4.23), (4.24) and (4.25) that

$$\delta_\lambda(\eta) = O_\eta(U^{-90}). \quad (4.26)$$

Taking $\alpha_0 = 2$ in (2.12), we have

$$\Theta(nx) = \sqrt{2}(\sqrt{nx})^{-1/2} \sum_{\alpha=0}^2 \frac{\gamma_\alpha}{(4\pi)^\alpha} (\sqrt{nx})^{-\alpha} \cos(4\pi\sqrt{nx} + \frac{\pi}{4} + \frac{\alpha\pi}{2}) + O((\sqrt{nx})^{-7/2}). \quad (4.27)$$

By (4.26) and (4.27) and putting $x = x_1^2$ in the integral (4.10), we have

$$S_\lambda^*(\eta) = \&_\lambda^+(\eta) + \&_\lambda^-(\eta) + O(\varepsilon_1) + O_\eta(U^{-\frac{3}{2}}), \quad (4.28)$$

where

$$\&_\lambda^\pm(\eta) = \sqrt{2}e(-\frac{1}{8} \pm \frac{1}{8}) \sum_{n \leq N} d(n)n^{-\frac{1}{4}} \sum_{\alpha=0}^2 \gamma_\alpha \left(\frac{\pm i}{4\pi\sqrt{n}}\right)^\alpha j_{n\alpha}^\pm(\eta), \quad (4.29)$$

$$j_{n\alpha}^\pm(\eta) = \frac{W_\lambda(\eta)}{h_0} \int_0^{h_0} dh \int_{\sqrt{U}+h}^{(\lambda_0-\lambda)\sqrt{U}+h} \frac{G_\lambda(\sqrt{n}, \eta) e(x^2 + 2x(\pm\sqrt{n} - (\sqrt{U} + \eta + \nu))) dx}{x^\alpha}. \quad (4.30)$$

As $n > n_0 > 2U$, the integration by parts gives

$$\begin{aligned} j_{n\alpha}^\pm(\eta) &= \\ &= \frac{W_\lambda(\eta)}{h_0} \int_0^{h_0} \frac{G_\lambda(x+h, \eta) e((x+h)^2 + 2(x+h)(\pm\sqrt{n} - \sqrt{U} - \eta - \nu))}{(x+h)^\alpha 4\pi i (x+h \pm \sqrt{n} - \sqrt{U} - \eta - \nu)} \Big|_{\sqrt{U}}^{(\lambda_0-\lambda)\sqrt{U}} \\ &\quad - \frac{W_\lambda(\eta)}{4\pi i h_0} \int_0^{h_0} \left(\frac{G_\lambda(x+h, \eta)}{(x+h)^\alpha} \right)'_x \frac{e((x+h)^2 + 2(x+h)(\pm\sqrt{n} - \sqrt{U} - \eta - \nu)) dx}{(x+h \pm \sqrt{n} - \sqrt{U} - \eta - \nu)} \\ &\ll \frac{U}{n}, \end{aligned}$$

for $h_0 = \frac{1}{\sqrt{UL}^2}$, where the last inequality is given by integrating by parts for h . Since

$$\sum_{n > n_0} d(n)n^{-5/4}U \ll \frac{UL}{n_0^{1/4}},$$

then taking $n_0 = U^{20}$, by (4.29) we obtain

$$\&_\lambda^\pm(\eta) = \sqrt{2}e(-\frac{1}{8} \pm \frac{1}{8}) \sum_{n \leq U^{20}} d(n)n^{-\frac{1}{4}} \sum_{\alpha=0}^2 \gamma_\alpha \left(\frac{\pm i}{4\pi\sqrt{n}}\right)^\alpha j_{n\alpha}^\pm(\eta) + O_\eta(U^{-\frac{3}{2}}). \quad (4.31)$$

As $\sqrt{n} > (\lambda_0 - \lambda - 1)\sqrt{U}$, moving the integral of $j_{n\alpha}^-(\eta)$, we obtain

$$j_{n\alpha}^-(\eta) = \frac{1}{h_0} \int_0^{h_0} dh \left(\int_{\sqrt{U}+h}^{\sqrt{U}+h-z_0} + \int_{\sqrt{U}+h-z_0}^{(\lambda_0-\lambda)\sqrt{U}+h-z_0} + \int_{(\lambda_0-\lambda)\sqrt{U}+h-z_0}^{(\lambda_0-\lambda)\sqrt{U}+h} \right)$$

$$\begin{aligned}
& \times \frac{e(z^2 - 2z(\sqrt{U} + \sqrt{n} + \nu))f(z, \eta)dz}{z^\alpha} \\
& = \frac{1}{h_0} \int_0^{h_0} dh (I_{1nh}^-(\eta) + I_{2nh}^-(\eta) + I_{3nh}^-(\eta)), \quad \text{say}
\end{aligned} \tag{4.32}$$

where $z_0 = Le(\frac{1}{8})$ and

$$f(z, \eta) = W_\lambda(\eta)G_\lambda(z, \eta)e(-2z\eta). \tag{4.33}$$

For $z = x - \rho e(\frac{1}{8})$, we have

$$|e(z^2 - 2\sqrt{n}(\sqrt{U} + \sqrt{n} + \nu)z)| = e^{-2\pi\rho^2 - 2\sqrt{2}\pi\rho(\sqrt{U} + \sqrt{n} + \nu - x)} \ll e^{-2\pi\rho^2} \tag{4.34}$$

holds for $x \leq (\lambda_0 - \lambda)\sqrt{U} + h$, $\sqrt{n} > (\lambda_0 - \lambda - 1)\sqrt{U}$, $h, \nu = O(L^{-1})$, $0 \leq \rho \leq L$. As $z \in I_{1nh}^-(\eta)$, $z = \sqrt{U} + h - \rho e(\frac{1}{8})$; as $z \in I_{3nh}^-(\eta)$, $z = (\lambda_0 - \lambda)\sqrt{U} + h - \rho e(\frac{1}{8})$. Repeating the proof of (4.26), we have

$$f(z, \eta) = O_\eta(U^{-90}), \quad z \in I_{1nh}^-, \quad I_{3nh}^-. \tag{4.35}$$

It follows from (4.34), (4.35) and $z^{-\alpha} \leq 1$ that

$$I_{1nh}^-(\eta) + I_{3nh}^-(\eta) = O_\eta(U^{-90}). \tag{4.36}$$

As $z \in I_{2nh}^-(\eta)$, $z = x - Le(\frac{1}{8})$. Clearly, $f(z, \eta) \ll 1$, $z = O(L)$. Hence, by (4.33) and the definition (4.32) of $I_{2nh}^-(\eta)$ we have

$$I_{2nh}^-(\eta) \ll \int_{\sqrt{U}+h}^{(\lambda_0-\lambda)\sqrt{U}+h} e^{-2\pi L^2} dx \ll e^{-L^2} \ll_\eta U^{-90}. \tag{4.37}$$

Moreover, (4.32), (4.36) and (4.37) deduce that

$$j_{n\alpha}^-(\eta) = O_\eta(U^{-90}) \tag{4.38}$$

for $\sqrt{n} > (\lambda_0 - \lambda - 1)\sqrt{U}$. Moving the integral line of $j_{n\alpha}^+(\eta)$, we obtain

$$\begin{aligned}
j_{n\alpha}^+(\eta) &= \frac{1}{h_0} \int_0^{h_0} dh \left(\int_{\sqrt{U}+h}^{\sqrt{U}+h+z_0} + \int_{\sqrt{U}+h+z_0}^{(\lambda_0-\lambda)\sqrt{U}+h+z_0} + \int_{(\lambda_0-\lambda)\sqrt{U}+h+z_0}^{(\lambda_0-\lambda)\sqrt{U}+h} \right) \\
&\quad \times \frac{e(z^2 + 2(-\sqrt{U} + \sqrt{n} - \nu)z)f(z, \eta)dz}{z^\alpha} \\
&= \frac{1}{h_0} \int_0^{h_0} dh (I_{1nh}^+(\eta) + I_{2nh}^+(\eta) + I_{3nh}^+(\eta)), \quad \text{say}
\end{aligned} \tag{4.39}$$

If $z = x + \rho e(\frac{1}{8})$, then

$$\begin{aligned}
|e(z^2 + 2(\sqrt{U} + \sqrt{n} - \nu)z)| &= |e(i\rho^2 + 2\rho e(\frac{1}{8})(-\sqrt{U} + \sqrt{n} - \nu + x))| \\
&= e^{-2\pi\rho^2 - 2\sqrt{2}\rho(-\sqrt{U} + \sqrt{n} - \nu + x)}
\end{aligned}$$

$$\ll e^{-2\pi\rho^2} \quad (4.40)$$

for $x \geq \sqrt{U} + h, n \geq 1$, and $\nu, h = O(L^{-1}), 0 \leq \rho \leq L$.

Similar to the proof of (4.38), we have

$$j_{n\alpha}^+(\eta) = O_\eta(U^{-90}) \quad (4.41)$$

for $n \geq 1$. It follows from (4.28), (4.31), (4.38) and (4.41) that

$$S_\lambda^*(\eta) = \sqrt{2}e(-\frac{1}{4}) \sum_{\sqrt{n} \leq (\lambda_0 - \lambda - 1)\sqrt{U}} d(n)n^{-\frac{1}{4}} \sum_{\alpha=0}^2 \gamma_\alpha(\frac{-i}{4\pi\sqrt{n}})^\alpha j_{n\alpha}^-(\eta) + \delta(\eta) + O_\eta(U^{-\frac{3}{2}}), \quad (4.42)$$

where

$$\delta(\eta) \ll_\eta \sum_{n \leq U^{20}} d(n)n^{-\frac{1}{4}}U^{-90} \ll_\eta U^{-\frac{3}{2}}. \quad (4.43)$$

Moving the inner integral line of $j_{n\alpha}^-(\eta)$, we have

$$\begin{aligned} j_{n\alpha}^-(\eta) &= \frac{1}{h_0} \int_0^{h_0} dh \left(\int_{\sqrt{U}+h}^{\sqrt{U}+h-z_0} + \int_{\sqrt{U}+h-z_0}^{x_n-z_0} + \int_{x_n-z_0}^{x_n+z_0} + \int_{x_n+z_0}^{(\lambda_0-\lambda)\sqrt{U}+h+z_0} + \int_{(\lambda_0-\lambda)\sqrt{U}+h+z_0}^{(\lambda_0-\lambda)\sqrt{U}+h} \right) \\ &\quad \times \frac{W_\lambda(\eta)e((z-x_n)^2 - x_n^2)dz}{z^\alpha} \\ &= \frac{1}{h_0} \int_0^{h_0} dh (I_{-1nh}^-(\eta) + I_{-2nh}^-(\eta) + I_0(\eta) + I_{1nh}^+(\eta) + I_{2nh}^+(\eta)), \quad \text{say,} \end{aligned} \quad (4.44)$$

for $\sqrt{n} \leq (\lambda_0 - \lambda - 1)\sqrt{U}$, where $z_0 = Le(\frac{1}{8})$,

$$x_n = \sqrt{U} + \sqrt{n} + \nu + \eta. \quad (4.45)$$

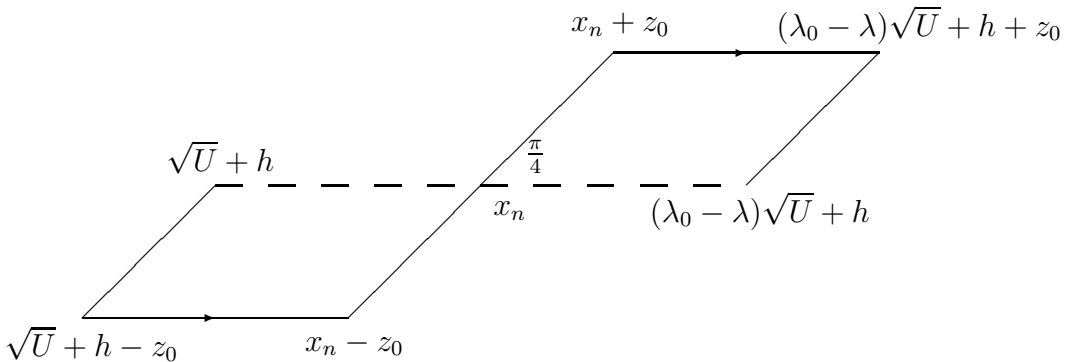


Fig.4.1

Repeating the proof of (4.38) and (4.41), we have

$$I_{-2nh}^-(\eta) + I_{-1nh}^-(\eta) + I_{1nh}^+(\eta) + I_{2nh}^+(\eta) \ll_\eta U^{-90}. \quad (4.46)$$

Hence

$$\begin{aligned}
j_{n\alpha}^-(\eta) &= \frac{1}{h_0} \int_0^{h_0} I_0(\eta) dh + O_\eta(U^{-90}) = I_0(\eta) + O_\eta(U^{-90}) \\
&= W_\lambda(\eta) e(-x_n^2) \int_{x_n - z_0}^{x_n + z_0} \frac{G_\lambda(z, \eta) e((z - x_n)^2) dz}{z^\alpha} + O_\eta(U^{-90}) \\
&= W_\lambda(\eta) e(-x_n^2 + \frac{1}{8}) \int_{-L}^L \frac{G_\lambda(x_n + \rho e(\frac{1}{8}), \eta) e^{-2\pi\rho^2} d\rho}{(x_n + \rho e(\frac{1}{8}))^\alpha} + O_\eta(U^{-90}). \tag{4.47}
\end{aligned}$$

By (4.1) and (4.2),

$$\begin{aligned}
G_\lambda(x_n + \rho e(\frac{1}{8}), \eta) &= e^{-\frac{4\pi(x_n + \rho e(\frac{1}{8}) + \lambda(\sqrt{U} + \eta + \nu))^2}{A^2}} \\
&= G_{\lambda+1}(\sqrt{n}, \eta) (1 + C_1(n)\rho + C_2(n)\rho^2 + C_3(n)\rho^3 + O(\frac{\rho^4}{U^2})), \tag{4.48} \\
C_1(n) &\leq \frac{1}{\sqrt{U}}, \quad C_3(n) \ll \frac{1}{\sqrt{U^3}}, \\
C_2(n) &\ll \frac{1}{U}, \quad C'_2(x) \ll \frac{1}{U\sqrt{xU}}
\end{aligned}$$

and

$$\begin{aligned}
W_\lambda(\eta) e(-x_n^2) &= e(-(\lambda + 1)(\sqrt{U} + \eta + \nu)^2 - 2\sqrt{n}(\sqrt{U} + \eta + \nu) - n) \\
&= W_{\lambda+1}(\eta) e(-2\sqrt{n}(\sqrt{U} + \eta + \nu))
\end{aligned}$$

for $e(-n) = 1$. Therefore,

$$\begin{aligned}
j_{n\alpha}^-(\eta) &= \frac{e(\frac{1}{8}) W_{\lambda+1}(\eta) G_{\lambda+1}(\sqrt{n}, \eta) e(-2\sqrt{n}(\sqrt{U} + \eta + \nu))}{(\sqrt{U} + \sqrt{n} + \nu + \eta)^\alpha} \\
&\quad \times \int_{-L}^L e(-2\pi\rho^2) (1 + C_1(n)\rho + C_2(n)\rho^2 + C_3(n)\rho^3 + O(\frac{\rho^4}{U^2})) (1 + O(\frac{\alpha|\rho|}{(\sqrt{U})^\alpha})) \\
&= \frac{e(\frac{1}{8}) W_{\lambda+1}(\eta) G_{\lambda+1}(\sqrt{n}, \eta) e(-2\sqrt{n}(\sqrt{U} + \eta + \nu))}{\sqrt{2}(\sqrt{U} + \sqrt{n} + \nu + \eta)^\alpha} (1 + C_{20}(n) + O(\frac{1}{U^2}) + \frac{\alpha}{(\sqrt{U})^\alpha}) \tag{4.49}
\end{aligned}$$

where

$$C_{20}(n) \ll \frac{1}{U}, \quad C'_{20}(x) \ll \frac{1}{U\sqrt{xU}}. \tag{4.50}$$

It follows from (4.42) and (4.49) that

$$S_\lambda^*(\eta) = e(-\frac{1}{8}) W_{\lambda+1}(\eta) \sum_{\sqrt{n} \leq (\lambda_0 - \lambda - 1)\sqrt{U}} d(n) n^{-\frac{1}{4}} G_{\lambda+1}(\sqrt{n}, \eta)$$

$$\times \sum_{\alpha=0}^2 \gamma_{\alpha} \frac{(-i)^{\alpha}}{(4\pi\sqrt{n}(\sqrt{U} + \eta + \nu))^{\alpha}} e(-2\sqrt{n}(\sqrt{U} + \eta + \nu)) + \delta_1 + \delta_2 + O_{\eta}(U^{-\frac{3}{2}}), \quad (4.51)$$

where

$$\begin{aligned} \delta_1 &\ll \sum_{\sqrt{n} \leq (\lambda_0 - \lambda - 1)\sqrt{U}} d(n)n^{-\frac{1}{4}} \sum_{\alpha=0}^2 \frac{1}{(\sqrt{n} + \sqrt{U})^{\alpha}} \left(\frac{1}{U^2} + \frac{\alpha}{(\sqrt{U})^{\alpha}} \right) \\ &\ll U^{-3/4} \ll_{\eta} U^{-3/4+\varepsilon}, \quad 0 < \varepsilon = O\left(\frac{1}{\log L}\right), \end{aligned} \quad (4.52)$$

$$\begin{aligned} \delta_2 &= e\left(-\frac{1}{8}\right) W_{\lambda+1}(\eta) \sum_{\sqrt{n} \leq (\lambda_0 - \lambda - 1)\sqrt{U}} C_{20}(\sqrt{n}) d(n) n^{-\frac{1}{4}} G_{\lambda+1}(\sqrt{n}, \eta) \\ &\quad \times \sum_{\alpha=0}^2 \gamma_{\alpha} \frac{(-i)^{\alpha}}{(4\pi\sqrt{n}(\sqrt{U} + \sqrt{n} + \eta + \nu))^{\alpha}} e(-2\sqrt{n}(\sqrt{U} + \eta + \nu)) \\ &= e\left(-\frac{1}{8}\right) W_{\lambda+1}(\eta) \\ &\quad \times \sum_{\sqrt{n} \leq (\lambda_0 - \lambda - 1)\sqrt{U}} C_{20}(\sqrt{n}) d(n) n^{-\frac{1}{4}} e(-2\sqrt{n}(\sqrt{U} + \eta + \nu)) + O(U^{-3/4}). \end{aligned} \quad (4.53)$$

By (4.51) and Lemma 3.4,

$$\delta_2 \ll U^{-3/4+\varepsilon} \ll_{\eta} U^{-3/4+\varepsilon}, \quad 0 < \varepsilon = O\left(\frac{1}{\log L}\right). \quad (4.54)$$

The last inequality has been used the explain (1.32). Thus, by (4.51), if $\lambda = \lambda_0 - 2$, then

$$S_{\lambda-2}^*(\eta) = S_{\lambda_0-1}(\eta) + \delta_{\lambda_0-1}(\eta) + O_{\eta}(U^{-3/4+\varepsilon}); \quad (4.55)$$

if $0 \leq \lambda \leq \lambda_0 - 1$, then

$$S_{\lambda}^*(\eta) = S_{\lambda+1}(\eta) + S_{\lambda+1}^*(\eta) + \delta_{\lambda+1}(\eta) + O_{\eta}(U^{-3/4+\varepsilon}), \quad (4.56)$$

where $0 < \varepsilon = O\left(\frac{1}{\log L}\right)$ and

$$\begin{aligned} \delta_{\lambda+1}(\eta) &= \\ &= e\left(-\frac{1}{8}\right) W_{\lambda+1}(\eta) \sum_{\alpha=1}^2 \gamma_{\alpha} \left(\frac{-i}{4\pi}\right)^{\alpha} \sum_{\sqrt{U} < \sqrt{n} \leq (\lambda_0 - \lambda - 1)\sqrt{U}} d(n) n^{-\frac{1}{4} - \frac{\alpha}{2}} \frac{e(-2\sqrt{n}(\sqrt{U} + \eta + \nu))}{(\sqrt{U} + \sqrt{n} + \nu + \eta)^{\alpha}}. \end{aligned} \quad (4.57)$$

Taking $f(n) = \frac{1}{(\sqrt{U} + \sqrt{n} + \nu + \eta)^{\alpha}}$ in Lemma 3.5, we know that the inner sum is $\ll U^{-3/4+\varepsilon}$, $\alpha \geq 1$, that is

$$\delta_{\lambda+1}(\eta) \ll U^{-3/4+\varepsilon}, \quad 0 < \varepsilon = O\left(\frac{1}{\log L}\right). \quad (4.58)$$

Finally, (4.6) follows from (4.55), (4.56) and (4.58) and the proof of Lemma 4.1 is complete. \square

Lemma 4.2 *Let*

$$\begin{aligned}
S_{\frac{1}{2}}(\eta) &= \sum_{\frac{\sqrt{U}}{2} < \sqrt{n} \leq \lambda_0 \sqrt{U}} d(n) n^{-\frac{1}{4}} e^{-\frac{4\pi n}{A^2}} e(-2\sqrt{n}(\sqrt{U} + \eta + \nu) - \frac{1}{8}) \\
&= S_0^*(\eta) + \sum_{\frac{\sqrt{U}}{2} < \sqrt{n} \leq \sqrt{U}} d(n) n^{-\frac{1}{4}} e^{-\frac{4\pi n}{A^2}} e(-2\sqrt{n}(\sqrt{U} + \eta + \nu) - \frac{1}{8}), \tag{4.59}
\end{aligned}$$

then

$$S_{\frac{1}{2}}(\eta) = \sum_{\lambda=1}^{\lambda_0} S_{\lambda}^+(\eta) + \sum_{\lambda=1}^{\lambda_0} S_{\lambda}^-(\eta) + O_{\eta}(U^{-3/4+\varepsilon}), \quad 0 < \varepsilon = O\left(\frac{1}{\log L}\right), \tag{4.60}$$

where $S_0^*(\eta)$, η , ν are as Lemma 4.1, and

$$\begin{aligned}
S_{\lambda}^+(\eta) &= W_{\lambda}(\eta) \sum_{\sqrt{n} \leq \frac{\sqrt{U}}{2}} d(n) n^{-\frac{1}{4}} G_{\lambda}(\sqrt{n}, \eta) \\
&\quad \times \sum_{\alpha=0}^2 \gamma_{\alpha} \frac{(-i)^{\alpha} e(-2\sqrt{n}(\sqrt{U} + \eta + \nu) - \frac{1}{8})}{(4\pi\sqrt{n}(\sqrt{U} + \sqrt{n} + \eta + \nu))^{\alpha}}, \tag{4.61}
\end{aligned}$$

$$\begin{aligned}
S_{\lambda}^-(\eta) &= W_{\lambda}(\eta) \sum_{\sqrt{n} \leq \frac{\sqrt{U}}{2}} d(n) n^{-\frac{1}{4}} G_{\lambda}(-\sqrt{n}, \eta) \\
&\quad \times \sum_{\alpha=0}^2 \gamma_{\alpha} \frac{(i)^{\alpha} e(2\sqrt{n}(\sqrt{U} + \eta + \nu) + \frac{1}{8})}{(4\pi\sqrt{n}(\sqrt{U} - \sqrt{n} + \eta + \nu))^{\alpha}}. \tag{4.62}
\end{aligned}$$

Proof: Sinc

$$\begin{aligned}
S_{\lambda}(\eta) &= W_{\lambda}(\eta) \sum_{\sqrt{n} \leq \sqrt{U}} d(n) n^{-\frac{1}{4}} G_{\lambda}(\sqrt{n}, \eta) \\
&\quad \times \sum_{\alpha=0}^2 \gamma_{\alpha} \frac{(-i)^{\alpha} e(-2\sqrt{n}(\sqrt{U} + \eta + \nu) - \frac{1}{8})}{(4\pi\sqrt{n}(\sqrt{U} + \sqrt{n} + \eta + \nu))^{\alpha}} \\
&= W_{\lambda}(\eta) \left(\sum_{\sqrt{n} \leq \frac{\sqrt{U}}{2}} + \sum_{\frac{\sqrt{U}}{2} < \sqrt{n} \leq \sqrt{U}} \right),
\end{aligned}$$

then by Lemma 4.1,

$$S_{\frac{1}{2}}(\eta) = \sum_{\lambda=1}^{\lambda_0} S_{\lambda}^+(\eta) + \sum_{\lambda=0}^{\lambda_0-1} H_{\lambda}(\eta) + O_{\eta}(U^{-3/4+\varepsilon}), \tag{4.63}$$

where $S_{\lambda_0}^+(\eta) \ll e^{-L^2}$ is used, and

$$\begin{aligned}
H_\lambda(\eta) &= W_\lambda(\eta) \sum_{\frac{\sqrt{U}}{2} < \sqrt{n} \leq \sqrt{U}} d(n) n^{-\frac{1}{4}} G_\lambda(\sqrt{n}, \eta) \\
&\quad \times \sum_{\alpha=0}^2 \gamma_\alpha \frac{(-i)^\alpha e(-2\sqrt{n}(\sqrt{U} + \eta + \nu) - \frac{1}{8})}{(4\pi\sqrt{n}(\sqrt{U} + \sqrt{n} + \eta + \nu))^\alpha} \\
&= W_\lambda(\eta) \sum_{\frac{\sqrt{U}}{2} < \sqrt{n} \leq \sqrt{U}} d(n) n^{-\frac{1}{4}} G_\lambda(\sqrt{n}, \eta) e(-2\sqrt{n}(\sqrt{U} + \eta + \nu) - \frac{1}{8}) \\
&\quad + O(U^{-3/4+\varepsilon}),
\end{aligned} \tag{4.64}$$

the O -term is given by the proof of (4.58).

The following proof is similar to Lemma 4.1. Since

$$0 \leq \{(\frac{\sqrt{U}}{2})^2\} \leq \frac{3}{4}, (\frac{\sqrt{U}}{2} + h)^2 - \frac{\sqrt{U}}{2} \leq \frac{1}{L^{3/2}}$$

for $0 < h \leq \frac{1}{\sqrt{U}L^2}$, then we have the corresponding (4.9). If replace \sqrt{U} by $\sqrt{U}/2$ in (4.18), we have also

$$e(\pm 2\eta(\frac{\sqrt{U}}{2})) = 1 \tag{4.65}$$

for $\eta = \eta_1 + \dots + \eta_K, \eta_j = 0$ or k_1/\sqrt{U} . Similarly, we obtain that

$$H_\lambda(\eta) = \sqrt{2}e(\frac{1}{8}) \sum_{\sqrt{n} \leq \frac{\sqrt{U}}{2}} d(n) n^{-\frac{1}{4}} \sum_{\alpha=0}^2 \gamma_\alpha (\frac{i}{4\pi})^\alpha j_{n\alpha}^+(\eta) + O_\eta(U^{-3/2}), \tag{4.66}$$

(different from (4.42), present main term is $j_{n\alpha}^+$, because $x \in (\frac{\sqrt{U}}{2}, \sqrt{U} + h)$) where

$$\begin{aligned}
j_{n\alpha}^+(\eta) &= \frac{W_\lambda(\eta)}{h_0} \int_0^{h_0} dh \int_{\frac{\sqrt{U}}{2}+h}^{\sqrt{U}+h} \frac{G_\lambda(x, \eta) e(x^2 + 2x(\sqrt{n} - \sqrt{U} - \eta - \nu)) dx}{x^\alpha} \\
&= \frac{W_\lambda(\eta) e(-x_n^2)}{h_0} \int_0^{h_0} dh \int_{\frac{\sqrt{U}}{2}+h}^{\sqrt{U}+h} \frac{G_\lambda(x, \eta) e((x - x_n)^2) dx}{x^\alpha},
\end{aligned} \tag{4.67}$$

$$x_n = \sqrt{U} + \eta + \nu - \sqrt{n}. \tag{4.68}$$

Repeating the proof of (4.51) we see that

$$\begin{aligned}
H_\lambda(\eta) &= e(\frac{1}{8}) W_\lambda(\eta) \sum_{\sqrt{n} \leq \frac{\sqrt{U}}{2}} d(n) n^{-\frac{1}{4}} e(-x_n^2) G_\lambda(x_n, \eta) \sum_{\alpha=0}^2 \gamma_\alpha (\frac{i}{4\pi})^\alpha \frac{1}{(4\pi\sqrt{n}x_n)^\alpha} \\
&\quad + O_\eta(U^{-3/4+\varepsilon}).
\end{aligned} \tag{4.69}$$

Since

$$G_\lambda(x_n, \eta) = e^{-\frac{4\pi(\sqrt{U}+\eta+\nu-\sqrt{n}+\lambda(\sqrt{U}+\eta+\nu))^2}{A^2}} = G_{\lambda+1}(-\sqrt{n}, \eta) \tag{4.70}$$

$$\begin{aligned}
W_\lambda(\eta)e(-x_n^2) &= e(-(\lambda+1)(\sqrt{U} + \eta + \nu)^2 + 2\sqrt{n}(\sqrt{U} + \eta + \nu) + n) \\
&= W_{\lambda+1}(\eta)e(2\sqrt{n}(\sqrt{U} + \eta + \nu)),
\end{aligned} \tag{4.71}$$

then by (4.69),

$$H_\lambda(\eta) = S_{\lambda+1}^-(\eta) + O(U^{-3/4+\varepsilon}), \tag{4.72}$$

where $\varepsilon = O(\frac{1}{\log L})$ and $S_\lambda^-(\eta)$ is (4.62). Thus, we obtain (4.60) immediately by (4.63) and (4.72). The proof of Lemma 4.2 is complete. \square

Lemma 4.3 *Let*

$$\lambda_0 = b_0 = [C_0^2 L], \quad C_0 \geq 200, \tag{4.73}$$

where η is (1.15). If $\frac{1}{8} \leq \{2\sqrt{U}\nu\} \leq \frac{3}{8}$, then

$$\begin{aligned}
\operatorname{Re} S_{\frac{1}{2}}(\eta) &= -\operatorname{Re} \sum_{\sqrt{n} \leq \frac{\sqrt{U}}{2}} d(n)n^{-\frac{1}{4}} e^{-\frac{4\pi n}{A^2}} e(-2\sqrt{n}(\sqrt{U} + \eta + \nu) - \frac{1}{8}) \\
&\quad \times \sum_{\alpha=0}^2 \gamma_\alpha \left(\frac{-i}{4\pi\sqrt{n}(\sqrt{n} + \sqrt{U} + \eta + \nu)} \right)^\alpha + O_\eta(U^{-3/4+\varepsilon}),
\end{aligned} \tag{4.74}$$

where $S_{\frac{1}{2}}(\eta)$ is (4.59), $\varepsilon = O(\frac{1}{\log L})$.

Proof: By the definitions (4.1) and (4.2) of $G_\lambda(\sqrt{n}, \eta)$ and $W_\lambda(\eta)$,

$$G_0(\sqrt{n}, \eta) = e^{-\frac{4\pi n}{A^2}}, \tag{4.75}$$

$$W_0(\eta) = 1. \tag{4.76}$$

By Lemma 4.2,

$$S_{\frac{1}{2}}(\eta) = \sum_{\lambda=1}^{b_0} (S_\lambda^+(\eta) + S_\lambda^-(\eta)) + O_\eta(U^{-3/4+\varepsilon}), \quad \varepsilon = O(\frac{1}{\log L}). \tag{4.77}$$

Since

$$\frac{1}{(\sqrt{n}(\sqrt{U} + \eta + \nu - \sqrt{n}))^\alpha} = \frac{1}{(\sqrt{n}(\sqrt{U} + \eta + \nu + \sqrt{n}))^\alpha} + (\sqrt{n})^{1-\alpha} \delta_\alpha(n), \tag{4.78}$$

where

$$\delta_\alpha(n) = \frac{1}{\sqrt{n}} \left(\frac{1}{(\sqrt{U} + \eta + \nu - \sqrt{n})^\alpha} - \frac{1}{(\sqrt{U} + \eta + \nu + \sqrt{n})^\alpha} \right) \ll \frac{\alpha}{(\sqrt{U})^{\alpha+1}}, \tag{4.79}$$

$$\delta'_\alpha(x) \ll \frac{\alpha}{\sqrt{xU}(\sqrt{U})^{\alpha+1}}, \tag{4.80}$$

moreover, $\delta_\alpha(n) = 0$ for $\alpha = 0$. By Lemma 3.5 ,

$$\begin{aligned}
\sum_{\alpha=1}^2 \gamma_\alpha \left(\frac{i}{4\pi} \right)^\alpha \sum_{\sqrt{n} \leq \frac{\sqrt{U}}{2}} d(n)n^{-\frac{1}{4}} (\sqrt{n})^{1-\alpha} \delta_\alpha(n) G_\lambda(-\sqrt{n}, \eta) e(2\sqrt{n}(\sqrt{U} + \eta + \nu)) &\ll U^{-3/4+\varepsilon}.
\end{aligned} \tag{4.81}$$

This and (4.62) give

$$S_{\lambda}^{-}(\eta) = W_{\lambda}(\eta) \sum_{\sqrt{n} \leq \frac{\sqrt{U}}{2}} d(n) n^{-\frac{1}{4}} G_{\lambda}(-\sqrt{n}, \eta) \sum_{\alpha=0}^2 \gamma_{\alpha} e(2\sqrt{n}(\sqrt{U} + \eta + \nu) + \frac{1}{8}) \\ \times \left(\frac{i}{4\pi\sqrt{n}(\sqrt{n} + \sqrt{U} + \eta + \nu)} \right)^{\alpha} + O(U^{-3/4+\varepsilon}), \quad \varepsilon = O\left(\frac{1}{\log L}\right). \quad (4.82)$$

It follows from (4.77), (4.82) and (4.61) that

$$S_{\frac{1}{2}}(\eta) + \overline{S_{\frac{1}{2}}(\eta)} = \sum_{\sqrt{n} \leq \sqrt{U}} \sum_{\alpha=0}^2 d(n) n^{-\frac{1}{4}} \gamma_{\alpha} \frac{1}{(4\pi\sqrt{n}(\sqrt{n} + \sqrt{U} + \eta + \nu))^{\alpha}} \\ \times \left((-i)^{\alpha} e(-2\sqrt{n}(\sqrt{U} + \eta + \nu) - \frac{1}{8}) N(n) \right. \\ \left. + (-i)^{\alpha} e(-2\sqrt{n}(\sqrt{U} + \eta + \nu) - \frac{1}{8}) N(n) \right) + O_{\eta}(U^{-3/4+\varepsilon}), \quad (4.83)$$

where

$$N(n) = \sum_{\lambda=1}^{b_0} W_{\lambda}(\eta) G_{\lambda}(\sqrt{n}, \eta) + \sum_{\lambda=1}^{b_0} \overline{W_{\lambda}(\eta)} G_{\lambda}(-\sqrt{n}, \eta). \quad (4.84)$$

Putting $\lambda = -\lambda'$ in the second sum, note

$$\overline{W_{-\lambda}(\eta)} = e(\overline{\lambda(\sqrt{U} + \eta + \nu)}) = W_{\lambda}(\eta), \quad (4.85)$$

$$G_{-\lambda}(-\sqrt{n}, \eta) = e^{-\frac{4\pi(-\sqrt{n}-\lambda(\sqrt{U}+\eta+\nu))^2}{A^2}} = G_{\lambda}(\sqrt{n}, \eta), \quad (4.86)$$

we have

$$N(n) = \sum_{\lambda=-b_0, \lambda \neq 0}^{b_0} W_{\lambda}(\eta) G_{\lambda}(\sqrt{n}, \eta) \\ = -W_0(\eta) G_0(\sqrt{n}, \eta) + \sum_{\lambda=-b_0}^{b_0} e^{-\frac{4\pi(\sqrt{n}+\lambda(\sqrt{U}+\eta+\nu))^2}{A^2}} e(-\lambda(\sqrt{U} + \eta + \nu)^2) \\ = -e^{-\frac{4\pi n}{A^2}} + \sum_{\lambda=-\infty}^{\infty} e^{-\frac{\pi(\lambda+b_n)^2}{V_1}} e(-\lambda(\sqrt{U} + \eta + \nu)^2) + O(U^{-10}),$$

where

$$V_1 = \frac{A^2}{(\sqrt{U} + \eta + \nu)^2}, \quad b_n = \frac{\sqrt{n}}{\sqrt{U} + \eta + \nu}. \quad (4.87)$$

Using

$$e(-\lambda(\sqrt{U} + \eta + \nu)^2) = e(-\lambda(U + 2\eta\sqrt{U} + 2\nu\sqrt{U} + (\eta + \nu)^2)) \\ = e(-\lambda(\{2\nu\sqrt{U}\} + (\eta + \nu)^2)),$$

for $U = [U]$, $\eta = k_1/\sqrt{U}$, where k_1 is an integer, we have

$$N(n) = -e^{-\frac{4\pi n}{A^2}} + \sum_{\lambda=-\infty}^{\infty} e^{-\frac{\pi}{V_1}(\lambda+b_n)^2} e(-\lambda(\{2\nu\sqrt{U}\} + (\eta + \nu)^2)) + O(U^{-10}).$$

By Lemma 3.2,

$$N(n) = \sqrt{V_1} \sum_{\lambda=-\infty}^{\infty} e^{-\pi V_1(\lambda - \{2\nu\sqrt{U}\} - (\eta + \nu)^2)} e(b_n(\{2\nu\sqrt{U}\} + (\eta + \nu)^2 - \lambda) - e^{-\frac{4\pi n}{A^2}} + O(U^{-10}).$$

Since $\frac{1}{8} \leq \{2\nu\sqrt{U}\} \leq \frac{3}{8}$, $(\eta + \nu)^2 = O(L^{-1})$ and

$$V_1 = \frac{A^2}{(\sqrt{U} + \eta + \nu)^2} = C_0^2 L(1 + o(1)) \geq \frac{200^2 L}{2},$$

then

$$\begin{aligned} N(n) &= -e^{-\frac{4\pi n}{A^2}} + O(\sqrt{L} e^{-\frac{\pi C_0^2 L}{200}}) \\ &= -e^{-\frac{4\pi n}{A^2}} + O(U^{-10}). \end{aligned} \tag{4.88}$$

By (4.83) and (4.88) we obtain (4.74). The proof is complete. \square

5 EVALUATION of $S(B)$

Now we evaluate $S(B)$ in (1.19), i.e.,

$$S(B) = A \int_{-\frac{5\sqrt{L}}{A}}^{\frac{5\sqrt{L}}{A}} e^{-\pi A^2 \theta^2} d\theta \sum_{\xi_1 < \sqrt{n} - B - \theta \leq \xi_2} d(n) n^{-\frac{1}{4}},$$

where $A, B, \xi_1, \xi_2, \dots$ are (1.10)-(1.18).

By Lemma 2.4 and taking $\alpha_0 = 2$ in (1.12),

$$\begin{aligned} S(B) &= \sum_{n \leq N} d(n) A \int_{-\frac{5\sqrt{L}}{A}}^{\frac{5\sqrt{L}}{A}} e^{-\pi A^2 \theta^2} d\theta \int_{(B+\xi_1+\theta)^2}^{(B+\xi_2+\theta)^2} x^{-\frac{1}{4}} \Theta(nx) dx + S_l(B) + O(\varepsilon_1) \\ &= 2\sqrt{2} \sum_{n \leq N} d(n) n^{-\frac{1}{4}} \sum_{\alpha=0}^2 \frac{\gamma_\alpha A}{(4\pi\sqrt{n})^\alpha} \int_{-\frac{5\sqrt{L}}{A}}^{\frac{5\sqrt{L}}{A}} e^{-\pi A^2 \theta^2} j_{n\alpha}(\theta) d\theta \\ &\quad + S_l(B) + O(U^{-\frac{3}{2}}) + O(\varepsilon_1) \end{aligned} \tag{5.1}$$

for any $\varepsilon_1 > 0$ as $N \geq N(\varepsilon_1)$, where

$$\begin{aligned} j_{n\alpha}(\theta) &= \int_{B+\xi_1+\theta}^{B+\xi_2+\theta} \frac{\cos(4\pi\sqrt{n}x + \frac{\pi}{4} + \frac{\alpha\pi}{2}) dx}{x^\alpha} \\ &= \operatorname{Re} e^{(\alpha/4 + 1/8)} \int_{\xi_1}^{\xi_2} \frac{e(2\sqrt{n}(B+x+\theta)) dx}{(B+x+\theta)^\alpha}, \end{aligned} \tag{5.2}$$

$$S_l(B) = A \int_{-\frac{5\sqrt{L}}{A}}^{\frac{5\sqrt{L}}{A}} e^{-\pi A^2 \theta^2} d\theta \int_{(B+\xi_1+\theta)^2}^{(B+\xi_2+\theta)^2} x^{\frac{-1}{4}} (\log x + 2\gamma) dx. \quad (5.3)$$

The inner integral of $S_l(B)$ is

$$2 \int_{\xi_1+\theta}^{\xi_2+\theta} (B+x)^{\frac{1}{2}} (\log(B+x)^2 + 2\gamma) dx = \sum_{k=0}^{10} C_k \eta^k + O(U^{-9}).$$

Let $B' = B - \eta = \sqrt{U} + \frac{k+j}{2\sqrt{U}}$ (see 1.11). Then by (3.37),

$$S_l(B) = S_l(B' + \eta) = O_\eta(U^{-9}). \quad (5.4)$$

By (5.2), we have

$$\begin{aligned} j_{n\alpha}(\theta) &= \operatorname{Re} e(\alpha/4 + 1/8) \left(\frac{e(2\sqrt{n}(B+x+\theta))}{4\pi i \sqrt{n}(B+x+\theta)^\alpha} \right) \Big|_{x=\xi_1}^{x=\xi_2} \\ &\quad + O\left(\frac{1}{n} \left(1 + \int_{\xi_1}^{\xi_2} \frac{\alpha dx}{(B+x+\theta)^{\alpha+1}}\right)\right). \end{aligned}$$

The integration by parts for θ gives

$$A \int_{-\frac{5\sqrt{L}}{A}}^{\frac{5\sqrt{L}}{A}} e^{-\pi A^2 \theta^2} j_{n\alpha}(\theta) d\theta \ll \frac{A}{n}.$$

Therefore,

$$\sum_{U^{20} < n \leq N} d(n) n^{-\frac{1}{4}} \sum_{\alpha=0}^2 \frac{\gamma_\alpha A}{(4\pi \sqrt{n})^\alpha} \int_{-\frac{5\sqrt{L}}{A}}^{\frac{5\sqrt{L}}{A}} e^{-\pi A^2 \theta^2} j_{n\alpha}(\theta) d\theta \ll \sum_{U^{20} < n} d(n) n^{-\frac{5}{4}} A \ll U^{-4}. \quad (5.5)$$

By (5.1), (5.4) and (5.5) we obtain

$$S(B) = 2\sqrt{2} \sum_{n \leq U^{20}} d(n) n^{\frac{-1}{4}} \sum_{\alpha=0}^2 \frac{\gamma_\alpha A}{(4\pi \sqrt{n})^\alpha} \int_{-\frac{5\sqrt{L}}{A}}^{\frac{5\sqrt{L}}{A}} e^{-\pi A^2 \theta^2} j_{n\alpha}(\theta) d\theta + O_\eta(U^{-\frac{3}{2}}), \quad (\varepsilon_1 \rightarrow 0_+). \quad (5.6)$$

If $\sqrt{n} > C_0^2 L \sqrt{U}$, then

$$\begin{aligned} &\int_{-\frac{5\sqrt{L}}{A}}^{\frac{5\sqrt{L}}{A}} e^{-\pi A^2 \theta^2} j_{n\alpha}(\theta) d\theta \\ &= \operatorname{Re} e(\alpha/4 + 1/8) \int_{\xi_1}^{\xi_2} e(2\sqrt{n}(B+x)) dx A \int_{-\frac{5\sqrt{L}}{A}}^{\frac{5\sqrt{L}}{A}} e^{-\pi \theta^2} \frac{e(2\sqrt{n}\theta) d\theta}{(B+x+\theta)^\alpha}. \end{aligned} \quad (5.7)$$

Moving the inner integral line of (5.7), we have

$$A \left(\int_{-\frac{5\sqrt{L}}{A}}^{-\frac{5\sqrt{L}}{A} + \frac{i}{C_0 \sqrt{U}}} + \int_{-\frac{5\sqrt{L}}{A} + \frac{i}{C_0 \sqrt{U}}}^{\frac{5\sqrt{L}}{A} + \frac{i}{C_0 \sqrt{U}}} + \int_{\frac{5\sqrt{L}}{A} + \frac{i}{C_0 \sqrt{U}}}^{\frac{5\sqrt{L}}{A}} \right) e^{-\pi A^2 w^2} \frac{e(2\sqrt{n}w) dw}{(B+x+w)^\alpha} = I_1 + I_2 + I_3, \quad \text{say.} \quad (5.8)$$

As $w \in I_1$, $w = -\frac{5\sqrt{L}}{A} + iv$, $0 \leq v \leq \frac{1}{C_0\sqrt{U}}$, and

$$|e(2\sqrt{n}w)| = |e(2i\sqrt{n}v)| = e^{-4\pi\sqrt{n}v}. \quad (5.9)$$

Hence

$$\begin{aligned} I_1 &\ll A \int_0^{\frac{1}{C_0\sqrt{U}}} |e^{-\pi A^2(-\frac{5\sqrt{L}}{A}+iv)^2}| e^{-4\pi\sqrt{n}v} dv \\ &\ll A \int_0^{\frac{1}{C_0\sqrt{U}}} e^{-\pi C_0^2 L U (\frac{25L}{C_0^2 U L} - \frac{1}{C_0^4 \sqrt{U}})} dv \\ &\ll \frac{A}{\sqrt{U}} e^{-20\pi L} \ll U^{-20}. \end{aligned}$$

In the same way, $I_3 \ll U^{-20}$. When $w \in I_2$, $w = u + \frac{i}{C_0\sqrt{U}}$. By (5.9),

$$|e(2\sqrt{n}w)| = e^{\frac{-4\pi\sqrt{n}}{C_0\sqrt{U}}} \ll e^{-4\pi C_0 L} \ll U^{-40}, \text{ for } \sqrt{n} > C_0^2 L \sqrt{U}.$$

Hence

$$I_2 \ll U^{-20}.$$

Therefore,

$$\int_{-\frac{5\sqrt{L}}{A}}^{\frac{5\sqrt{L}}{A}} e^{-\pi A^2 \theta^2} j_{n\alpha}(\theta) d\theta \ll U^{-20} \quad (5.10)$$

for $\sqrt{n} > C_0^2 L \sqrt{U}$. If b_0 is (4.74), i.e.

$$b_0 = [C_0^2 L], \quad (5.11)$$

then

$$\begin{aligned} \sum_{(b_0+1/2)\sqrt{U} < \sqrt{n} \leq U^{10}} d(n) n^{\frac{-1}{4}} \sum_{\alpha=0}^2 \frac{\gamma_\alpha A}{(4\pi\sqrt{n})^\alpha} \int_{-\frac{5\sqrt{L}}{A}}^{\frac{5\sqrt{L}}{A}} e^{-\pi A^2 \theta^2} j_{n\alpha}(\theta) d\theta \\ \ll \sum_{(b_0+1/2)\sqrt{U} < \sqrt{n} \leq U^{10}} d(n) n^{\frac{-1}{4}} U^{-20} \ll U^{-3}. \end{aligned}$$

This and (5.6) give

$$S(B) = 2\sqrt{2} \sum_{\sqrt{n} \leq (b_0+1/2)\sqrt{U}} d(n) n^{\frac{-1}{4}} \sum_{\alpha=0}^2 \frac{\gamma_\alpha A}{(4\pi\sqrt{n})^\alpha} \int_{-\frac{5\sqrt{L}}{A}}^{\frac{5\sqrt{L}}{A}} e^{-\pi A^2 \theta^2} j_{n\alpha}(\theta) d\theta + O_\eta(U^{-1}), \quad (5.12)$$

where $j_{n\alpha}(\theta)$ is (5.2). By (5.12) and (5.2),

$$S(B) = 2\sqrt{2} \operatorname{Re} \int_{\xi_1}^{\xi_2} (T_0(B + \xi) + T_1(B + \xi)) d\xi + O_\eta(U^{-1}) \quad (5.13)$$

where ξ_1, ξ_2 are (1.18),

$$T_0(B + \xi) = e(1/8) \sum_{\sqrt{n} \leq \sqrt{U}/2} d(n) n^{\frac{-1}{4}} \sum_{\alpha=0}^2 \frac{\gamma_\alpha A i^\alpha}{(4\pi\sqrt{n})^\alpha} \int_{-\frac{5\sqrt{L}}{A}}^{\frac{5\sqrt{L}}{A}} e^{-\pi A^2 \theta^2} \frac{e(2\sqrt{n}(B + \xi + \theta)) d\theta}{(B + \xi + \theta)^\alpha}, \quad (5.14)$$

$$T_1(B + \xi) = e(1/8) \sum_{\sqrt{U}/2 < \sqrt{n} \leq (b_0+1/2)\sqrt{U}} d(n) n^{\frac{-1}{4}} A \int_{-\frac{5\sqrt{L}}{A}}^{\frac{5\sqrt{L}}{A}} e^{-\pi A^2 \theta^2} e(2\sqrt{n}(B + \xi + \theta)) d\theta + \delta_1(B + \xi) \quad (5.15)$$

for $e(\frac{\alpha}{4}) = i^\alpha$ with

$$\delta_1(B + \xi) = e(1/8) \sum_{\alpha=1}^2 \frac{\gamma_\alpha A i^\alpha}{(4\pi\sqrt{n})^\alpha} \times \int_{-\frac{5\sqrt{L}}{A}}^{\frac{5\sqrt{L}}{A}} e^{-\pi A^2 \theta^2} \cdot \sum_{\sqrt{U}/2 < \sqrt{n} \leq (b_0+1/2)\sqrt{U}} \frac{d(n) n^{\frac{-1}{4}} e(2\sqrt{n}(B + \xi + \theta)) d\theta}{(\sqrt{n}(B + x + \theta))^\alpha}. \quad (5.16)$$

Using Lemma 3.4 for the inner sum, we have

$$\delta_1(B + \xi) \ll U^{-1+\varepsilon}. \quad (5.17)$$

The inner integral of (5.15) is

$$A \int_{-\infty}^{\infty} e^{-\pi A^2 \theta^2} e(2\sqrt{n}(\sqrt{U} + \xi + \theta)) d\theta + O(e^{-L^2}) = e^{\frac{-4\pi n}{A^2}} + O(e^{-L^2}).$$

Hence

$$T_1(B + \xi) = e(1/8) \sum_{\sqrt{U}/2 < \sqrt{n} \leq (b_0+1/2)\sqrt{U}} d(n) n^{\frac{-1}{4}} e^{\frac{-4\pi n}{A^2}} e(2\sqrt{n}(B + \xi)) + O_\eta(U^{-1+\varepsilon}). \quad (5.18)$$

Therefore,

$$S(B) = 2\sqrt{2}\text{Re} \int_{\xi_1}^{\xi_2} (T_0(B + \xi) + T_1(B + \xi)) d\xi + O_\eta(U^{-1+\varepsilon}), \quad (5.19)$$

where $T_0(B + \xi)$ is (5.14), $T_1(B + \xi)$ is (5.18), ξ_1, ξ_2 are (1.18).

6 EVALUATION of $T_1(B + \xi)$

We are going to evaluate (5.18). For $0 < h \leq h_0 = 1/U$, $U = [X]$

$$0 < \left(\frac{\sqrt{U}}{2} + h\right)^2 - \frac{U}{4} = \sqrt{U}h + h^2 \leq \frac{1}{L},$$

$$0 \leq \left\{\frac{U}{4}\right\} \leq \frac{3}{4}.$$

Hence there doesn't exist any integer in the interval $((\frac{\sqrt{U}}{2})^2, (\frac{\sqrt{U}}{2} + h)^2]$. So that

$$\sum_{\sqrt{U}/2 < \sqrt{n} \leq \sqrt{U}/2 + h} = 0, \text{ for } 0 < h \leq h_0 = 1/U. \quad (6.1)$$

In the same way,

$$\sum_{(b_0+1/2)\sqrt{U} < \sqrt{n} \leq (b_0+1/2)\sqrt{U} + h} = 0. \quad (6.2)$$

It follows from (5.18) that

$$T_1(B + \xi) = \frac{e(1/8)}{h_0} \int_0^{h_0} dh \sum_{\sqrt{U}/2 < \sqrt{n} \leq (b_0+1/2)\sqrt{U} + h} d(n) n^{-\frac{1}{4}} e^{\frac{-4\pi n}{A^2}} e(2\sqrt{n}(B + \xi)) + O_\eta(U^{-1+\varepsilon}), \quad (6.3)$$

where $h_0 = 1/U$. By Lemma 2.3

$$T_1(B + \xi) = \sum_{n \leq N} d(n) \frac{e(1/8)}{h_0} \int_0^{h_0} dh \int_{(\sqrt{U}/2 + h)^2}^{((b_0+1/2)\sqrt{U} + h)^2} x^{-\frac{1}{4}} e^{\frac{-4\pi x}{A^2}} e(2\sqrt{x}(B + \xi)) \Theta(nx) dx + T_{1l}(B + \xi) + O(\varepsilon_1) \quad (6.4)$$

for any $\varepsilon_1 > 0$, as $N \geq N(\varepsilon_1)$, where $\Theta(nx)$ is (2.12) and

$$T_{1l}(B + \xi) = \frac{e(1/8)}{h_0} \int_0^{h_0} dh \int_{(\sqrt{U}/2 + h)^2}^{((b_0+1/2)\sqrt{U} + h)^2} x^{-\frac{1}{4}} e^{\frac{-4\pi x}{A^2}} e(2\sqrt{x}(B + \xi)) (\log x + 2\gamma) dx$$

$$= \frac{2e(1/8)}{h_0} \int_0^{h_0} dh \int_{\sqrt{U}/2 + h}^{(b_0+1/2)\sqrt{U} + h} x^{\frac{1}{2}} e^{\frac{-4\pi x^2}{A^2}} e(2x(B + \xi)) (\log x^2 + 2\gamma) dx. \quad (6.5)$$

Changing the inner integral line of (6.5), we have

$$T_{1l}(B + \xi) = \frac{2e(1/8)}{h_0}$$

$$\times \int_0^{h_0} dh \left(\int_{\sqrt{U}/2 + h}^{\sqrt{U}/2 + h + i} + \int_{\sqrt{U}/2 + h + i}^{(b_0+1/2)\sqrt{U} + h + i} + \int_{(b_0+1/2)\sqrt{U} + h + i}^{(b_0+1/2)\sqrt{U} + h} \right) e(2z(B + \xi)) f_l(z) dz$$

$$= \frac{2e(1/8)}{h_0} \int_0^{h_0} dh (I_{1h}(B + \xi) + I_{2h}(B + \xi) + I_{3h}(B + \xi)), \quad \text{say} \quad (6.6)$$

where

$$f_l(z) = z^{1/2} e^{\frac{-4\pi z^2}{A}} (\log z^2 + 2\gamma). \quad (6.7)$$

Furthermore,

$$\begin{aligned} I_{1h}(B + \xi) &= \int_0^i e(2(B + \xi)(\sqrt{U}/2 + h + z)) f_l(\sqrt{U}/2 + h + z) dz \\ &= \int_0^i e(2(B' + \xi)(\sqrt{U}/2 + h + z)) f_l(\sqrt{U}/2 + h + z) e(\eta(\sqrt{U} + 2h + 2z)) dz, \end{aligned}$$

where $B = B' + \eta$. By (1.14) and (1.15), $e(\eta\sqrt{U}) = 1$, hence, by Lemma 3.6

$$e(\eta(\sqrt{U} + 2h + 2z)) = e(\eta(2h + 2z)) = O_\eta(U^{-99}).$$

Therefore,

$$I_{1h}(B + \xi) \ll_\eta U^{-99} \int_0^1 |f(\sqrt{U}/2 + h + i\rho) e(2(B' + \xi)i\rho)| d\rho \ll_\eta U^{-20}.$$

In the same way,

$$I_{3h}(B + \xi) \ll_\eta U^{-20}.$$

When $z \in I_{2h}(B + \xi)$, $z = x + i$. Thus,

$$|e(2z(B + \xi)) f_l(z)| \ll \sqrt{U} L |e(2(B + \xi)i)| = \sqrt{U} L e^{-4\pi(B + \xi)}.$$

Hence

$$I_{2h}(B + \xi) \ll \sqrt{U} L e^{-4\pi(B + \xi)} \int_{\sqrt{U}/2+h}^{(b_0+1/2)\sqrt{U}+h} dx \ll U^{-21}.$$

Therefore, by (6.6),

$$T_{1l}(B + \xi) = O_\eta(U^{-19}). \quad (6.8)$$

Putting $x = x_1^2$ in the integral (6.4), taking $\alpha_0 = 2$ in (2.12), noticing $2 \cos 2\pi x = e(x) + e(-x)$, $e(\frac{\alpha}{4}) = i^\alpha$, by (6.4) and (6.8) we obtain

$$T_1(B + \xi) = T_1^+(B + \xi) + T_1^-(B + \xi) + O_\eta(U^{-19}) + O(\varepsilon_1), \quad (6.9)$$

where

$$T_1^\pm(B + \xi) = \sqrt{2} \sum_{n \leq N} d(n) n^{-1/4} \sum_{\alpha=0}^2 \frac{\gamma_\alpha(\pm i)^\alpha}{(4\pi\sqrt{n})^\alpha} j_{n\alpha}^\pm(B + \xi), \quad (6.10)$$

$$j_{n\alpha}^\pm(B + \xi) = \frac{e(1/8 \pm 1/8)}{h_0} \int_0^{h_0} dh \int_{\sqrt{U}/2+h}^{(b_0+1/2)\sqrt{U}+h} \frac{e^{\frac{-4\pi x^2}{A^2}} e(2x(B + \xi \pm \sqrt{n})) dx}{x^\alpha}. \quad (6.11)$$

If $\sqrt{n} > 2\sqrt{U}$, then

$$\begin{aligned} j_{n\alpha}^\pm(B + \xi) &= \frac{e(1/8 \pm 1/8)}{4\pi i h_0} \int_0^{h_0} dh \frac{e^{\frac{-4\pi(x+h)^2}{A^2}} e(2(x+h)(B + \xi \pm \sqrt{n}))}{(B + \xi \pm \sqrt{n})(x+h)^\alpha} \Big|_{x=\sqrt{U}/2}^{x=(b_0+1/2)\sqrt{U}} \\ &\quad + O\left(\frac{1}{(B + \xi \pm \sqrt{n})^2}\right) \\ &\ll \frac{1}{h_0(B + \xi \pm \sqrt{n})^2} \ll \frac{U}{n} \end{aligned} \quad (6.12)$$

Since

$$\sum_{U^{20} < n \leq N} d(n) n^{-1/4-\alpha/2} \frac{U}{n} \ll U^{-3}$$

for $\alpha \geq 0$, then by (6.10),

$$T_1^\pm(B + \xi) = \sqrt{2} \sum_{\sqrt{n} \leq U^{10}} d(n) n^{-1/4} \sum_{\alpha=0}^2 \frac{\gamma_\alpha(\pm i)^\alpha}{(4\pi\sqrt{n})^\alpha} j_{n\alpha}^\pm(B + \xi) + O(U^{-3}). \quad (6.13)$$

Let $z = x + iy$, we see that

$$|e(2(B + \xi \pm \sqrt{n}))| = e(2iy(B + \xi \pm \sqrt{n})) = e^{-4\pi y(B + \xi \pm \sqrt{n})}. \quad (6.14)$$

Using the path (6.6), repeating the proof of (6.8), we obtain

$$j_{n\alpha}^+(B + \xi) = O_\eta(U^{-19}), \text{ for } n \geq 1. \quad (6.15)$$

In the same way, if $\sqrt{n} \leq \sqrt{U} - L^2$, then the right side of (6.14) is

$$e^{-4\pi y(B + \xi - \sqrt{n})} \ll e^{-4\pi y L^2}, \quad y \geq 0.$$

The path (6.6) gives also

$$j_{n\alpha}^-(B + \xi) = O_\eta(U^{-19}), \text{ for } \sqrt{n} \leq \sqrt{U} - L^2. \quad (6.16)$$

If $\sqrt{n} > \sqrt{U} + L^2$, then moving the inner integral line of $j_{n\alpha}^-(B + \xi)$ to

$$\int_{\sqrt{U}/2+h}^{\sqrt{U}/2+h-i} + \int_{\sqrt{U}/2+h-i}^{(b_0+1/2)\sqrt{U}+h-i} + \int_{(b_0+1/2)\sqrt{U}+h-i}^{(b_0+1/2)\sqrt{U}+h},$$

noticing

$$|e(2(B + \xi - \sqrt{n})z)| = e^{-4\pi|y(B + \xi - \sqrt{n})|}$$

for $z = x + iy$, $y \leq 0$, we have also

$$j_{n\alpha}^-(B + \xi) \ll_\eta U^{-19}, \text{ for } \sqrt{n} \leq \sqrt{U} - L^2. \quad (6.17)$$

It follows from (6.9), (6.13), (6.15), (6.16) and (6.17) that

$$\begin{aligned} T_1(B + \xi) &= \sqrt{2} \sum_{-L^2 < \sqrt{n} - \sqrt{U} \leq L^2} d(n) n^{-1/4} \sum_{\alpha=0}^2 \frac{\gamma_\alpha(-i)^\alpha}{(4\pi\sqrt{n})^\alpha} j_{n\alpha}^-(B + \xi) \\ &\quad + \delta + O_\eta(U^{-2}) \quad (\varepsilon_1 \rightarrow 0_+), \end{aligned} \quad (6.18)$$

where $j_{n\alpha}^-(B + \xi)$ is (6.11) and

$$\delta \ll_\eta \sum_{\sqrt{n} \leq U^{10}} d(n) n^{-1/4} \sum_{\alpha=0}^2 \frac{U^{-19}}{(\sqrt{n})^\alpha} \ll_\eta U^{-2}. \quad (6.19)$$

Next, we evaluate $j_{n\alpha}^-(B + \xi)$, $\alpha = 1, 2$. Using the integration by parts, It is easy to know that

$$j_{n\alpha}^-(B + \xi) \ll \frac{1}{(\sqrt{U})^\alpha |B + \xi - \sqrt{n}|},$$

or,

$$\begin{aligned} j_{n\alpha}^-(B + \xi) &\ll \frac{1}{h_0} \int_0^{h_0} dh \int_{\sqrt{U}/2+h}^{(b_0+1/2)\sqrt{U}+h} \frac{dx}{(\sqrt{U})^\alpha} \\ &\ll \frac{b_0\sqrt{U}}{(\sqrt{U})^\alpha} \ll L(\sqrt{U})^{1-\alpha}. \end{aligned}$$

Hence

$$j_{n\alpha}^-(B + \xi) \ll \min\left(\frac{L}{(\sqrt{U})^{1-\alpha}}, \frac{1}{(\sqrt{U})^\alpha |B + \xi - \sqrt{n}|}\right).$$

Moreover,

$$\begin{aligned} &\sum_{\alpha=1}^2 \frac{\gamma_\alpha(-i)^\alpha}{(4\pi)^\alpha} \sum_{-L^2 < \sqrt{n} - \sqrt{U} \leq L^2} d(n)n^{-1/4-\alpha/2} j_{n\alpha}^-(B + \xi) \\ &\ll L \sum_{|B+\xi-\sqrt{n}| \leq \frac{1}{\sqrt{U}}} d(n)n^{-3/4} + \sum_{|B+\xi-\sqrt{n}| \geq \frac{1}{\sqrt{U}}, |\sqrt{n}-\sqrt{U}| \leq L^2} \frac{d(n)n^{-3/4}}{|(B+\xi)^2 - n|} \\ &\ll U^{-\frac{3}{4}+\varepsilon}, \quad 0 < \varepsilon = O\left(\frac{1}{\log L}\right) \end{aligned} \quad (6.20)$$

This and (6.18), (6.19), (6.11) give

$$\begin{aligned} T_1(B + \xi) &= \sqrt{2} \sum_{-L^2 < \sqrt{n} - \sqrt{U} \leq L^2} d(n)n^{-1/4} j_{n0}^-(B + \xi) + \delta + O_\eta(U^{-\frac{3}{4}+\varepsilon}) \\ &= \frac{\sqrt{2}}{h_0} \int_0^{h_0} dh \sum_{-L^2 < \sqrt{n} - \sqrt{U} \leq L^2} d(n)n^{-1/4} \int_{\sqrt{U}/2+h}^{(b_0+1/2)\sqrt{U}+h} e^{\frac{-4\pi x^2}{A^2}} e(2x(B + \xi - \sqrt{n})) dx \\ &\quad + O_\eta(U^{-\frac{3}{4}+\varepsilon}), \quad h_0 = \frac{1}{U}, \quad 0 < \varepsilon = O\left(\frac{1}{\log L}\right) \end{aligned} \quad (6.21)$$

By Lemma 3.4,

$$\sum_{-L^2 < \sqrt{n} - \sqrt{U} \leq L^2} d(n)n^{-1/4} e(-2\sqrt{n}x) \ll x^{1/4+\varepsilon} \ll U^{1/4+\varepsilon}.$$

Hence

$$\frac{1}{h_0} \int_0^{h_0} dh \left(\int_{\sqrt{U}/2+h}^{\sqrt{U}/2} + \int_{(b_0+1/2)\sqrt{U}}^{(b_0+1/2)\sqrt{U}+h} \right) e^{\frac{-4\pi x^2}{A^2}} \sum_{-L^2 < \sqrt{n} - \sqrt{U} \leq L^2} d(n)n^{-1/4} e(2x(B + \xi - \sqrt{n}))$$

$$\ll \frac{U^{1/4+\varepsilon}}{h_0} \int_0^{h_0} h dh \ll U^{-\frac{3}{4}+\varepsilon}, \quad 0 < \varepsilon = O\left(\frac{1}{\log L}\right). \quad (6.22)$$

It follows from (6.21) and (6.22) that

$$\begin{aligned} T_1(B+\xi) &= \sqrt{2} \sum_{-L^2 < \sqrt{n}-\sqrt{U} \leq L^2} d(n)n^{-1/4} \int_{\sqrt{U}/2}^{(b_0+1/2)\sqrt{U}} e^{-\frac{4\pi(x)^2}{A^2}} e(2x(B+\xi-\sqrt{n})) dx + O_\eta(U^{-\frac{3}{4}+\varepsilon}), \end{aligned} \quad (6.23)$$

where $b_0 = [C_0^2 L]$, $0 < \varepsilon = O(\frac{1}{\log L})$.

7 EVALUATION of SUM $\sum_{k=0}^m T_1(B+\xi)$

Let

$$B_1 = \sqrt{U} + \frac{j}{2\sqrt{U}} + \eta + \xi, \quad (7.1)$$

then by (1.10),

$$B + \xi = B_1 + \frac{k}{2\sqrt{U}}. \quad (7.2)$$

Moreover, denote

$$Q_1(B_1) = \sum_{k=0}^m T_1(B+\xi) = \sum_{k=0}^m T_1(B_1 + \frac{k}{2\sqrt{U}}), \quad (7.3)$$

where $T_1(B)$ is (6.27). In this section we are going to prove that

$$\begin{aligned} \operatorname{Re}(Q_1(B_1)) &= \operatorname{Re} \sum_{k=0}^m T_1(B_1 + \frac{k}{2\sqrt{U}}) \\ &= -\frac{\sqrt{U}}{\sqrt{2}} \sum_{0 < \sqrt{n}-B_1 \leq \frac{m}{2\sqrt{U}}} d(n)n^{-1/4} + \operatorname{Re} \delta_{Q_1}(B_1), \end{aligned} \quad (7.4)$$

where $\delta_{Q_1}(B_1) \ll U^{\frac{1}{4}+\varepsilon}$, $0 < \varepsilon = O(\frac{1}{\log L}) = O(\frac{1}{\log \log U})$.

Proof of (7.4): By (6.23), (7.1), (7.2) and (7.3),

$$\begin{aligned} Q_1(B_1) &= \sqrt{2} \sum_{-L^2 < \sqrt{n}-\sqrt{U} \leq L^2} d(n)n^{-1/4} \int_{\frac{\sqrt{U}}{2}}^{(b_0+1/2)\sqrt{U}} e^{-\frac{4\pi x^2}{A^2}} e(2(B_1 - \sqrt{n})x) \rho(x) dx \\ &\quad + O_\eta(U^{-\frac{1}{4}+\varepsilon}) \end{aligned} \quad (7.5)$$

for $m \ll \sqrt{U}L^{-2}$, where $0 < \varepsilon = O(\frac{1}{\log L})$ and

$$\rho(x) = \sum_{k=0}^m e(\frac{xk}{\sqrt{U}}). \quad (7.6)$$

By Lemma 3.3,

$$\begin{aligned} \rho(x) &= \frac{1}{2} + \frac{1}{2}e(\frac{xm}{\sqrt{U}}) + O(U^{-9}) + \sum_{b=-U^{10}}^{U^{10}} \int_0^m e((\frac{x}{\sqrt{U}} - b)u)du \\ &= \rho_0(x) + \rho_1(x) + \rho_2(x) + O(U^{-9}), \end{aligned} \quad (7.7)$$

where

$$\rho_0(x) = \sum_{b=1}^{b_0} \int_0^m e((\frac{x}{\sqrt{U}} - b)u)du, \quad (7.8)$$

$$\rho_1(x) = \frac{1}{2} + \frac{1}{2}e(\frac{xm}{\sqrt{U}}) = \frac{1}{2} \sum_{\nu=0,m} e(\frac{x\nu}{\sqrt{U}}), \quad (7.9)$$

$$\rho_2(x) = \sum_b ' \int_0^m e((\frac{x}{\sqrt{U}} - b)u)du = \sum_b ' \frac{e(\frac{xu}{\sqrt{U}})}{2\pi i(\frac{x}{\sqrt{U}} - b)} \Big|_{u=0}^{u=m} \quad (7.10)$$

for $e(-b\nu) = 1$, $\nu = 0, m$, and \sum' is for $U^{-10} \leq b \leq 0$, or $b_0 + 1 \leq b \leq U^{10}$. It follows from (7.5) and (7.7) that

$$Q_1(B_1) = Q_{10}(B_1) + Q_{11}(B_1) + Q_{12}(B_1) + O_\eta(U^{-1}), \quad (7.11)$$

where

$$Q_{1j}(B_1) = \sqrt{2} \sum_{-L^2 < \sqrt{n} - \sqrt{U} \leq L^2} d(n)n^{-1/4} J_j(n), \quad j = 0, 1, 2, \quad (7.12)$$

$$J_j(n) = \int_C e^{-\frac{4\pi z^2}{A^2}} e(2(B_1 - \sqrt{n})z) \rho_j(z) dz = \int_{C_A^+} = \int_{C_A^-}, \quad (7.13)$$

C is from $\frac{\sqrt{U}}{2}$ to $(b_0 + 1/2)\sqrt{U}$, C_A^\pm is the broken line joining the points $\frac{\sqrt{U}}{2}$, $\frac{\sqrt{U}}{2} \pm iA/2$, $(b_0 + 1/2)\sqrt{U} \pm iA/2$ and $(b_0 + 1/2)\sqrt{U}$.

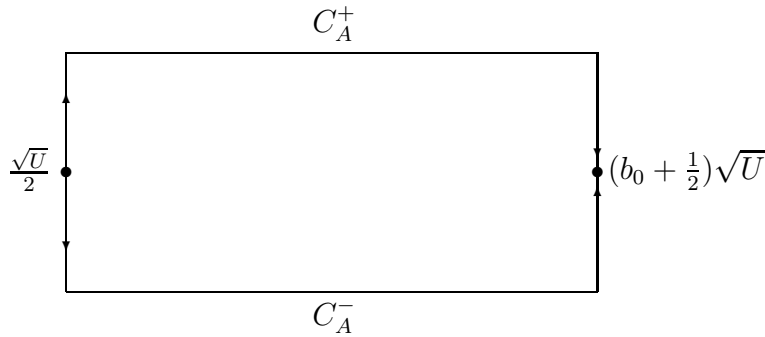


Fig. 7.1

By (7.10) and (7.13),

$$J_2(n) = \sum_b ' \int_C \frac{e^{-\frac{4\pi z^2}{A^2}} e(2(B_1 + \frac{u}{2\sqrt{U}} - \sqrt{n})z) dz}{2\pi i(\frac{z}{\sqrt{U}} - b)} \Big|_{u=0}^{u=m}. \quad (7.14)$$

It is obvious that the integrand is analytic for $\frac{\sqrt{U}}{2} \leq \operatorname{Re} z \leq (b_0 + 1/2)\sqrt{U}$, as $U^{-10} \leq b \leq 0$, or $b_0 + 1 \leq b \leq U^{10}$. For $z = x + iy$, $\frac{\sqrt{U}}{2} \leq x \leq (b_0 + 1/2)\sqrt{U}$, $-A/2 \leq y \leq A/2$, we have

$$|e^{-\frac{4\pi z^2}{A^2}}| = e^{-\frac{4\pi(x^2 - y^2)}{A^2}} \ll e^{-\frac{4\pi x^2}{A^2}}, \quad (7.15)$$

$$|e(2(B_1 + \frac{u}{2\sqrt{U}} - \sqrt{n})z)| = e^{-4\pi y(B_1 + \frac{u}{2\sqrt{U}} - \sqrt{n})}. \quad (7.16)$$

If $\sqrt{n} \leq B_1 + \frac{u}{2\sqrt{U}}$, then moving the integral line C to C_A^+ , we see that

$$\begin{aligned} \int_C &= \int_{C_A^+} = \left(\int_{\frac{\sqrt{U}}{2}}^{\frac{\sqrt{U}}{2} + \frac{iA}{2}} + \int_{\frac{\sqrt{U}}{2} + \frac{iA}{2}}^{(b_0+1/2)\sqrt{U} + \frac{iA}{2}} + \int_{(b_0+1/2)\sqrt{U} + \frac{iA}{2}}^{(b_0+1/2)\sqrt{U}} \right) \\ &\quad \times \frac{e^{-\frac{4\pi z^2}{A^2}} e(2(B_1 + \frac{u}{2\sqrt{U}} - \sqrt{n})z) dz}{2\pi i(\frac{z}{\sqrt{U}} - b)} \\ &= I_1 + I_2 + I_3, \quad \text{say} \end{aligned} \quad (7.17)$$

Clearly, for $b \leq 0$ or $b \geq b_0 + 1$ and

$$\frac{1}{\frac{z}{\sqrt{U}} - b} \ll \frac{1}{|b| + 1}, \quad z \in C_A^\pm, \quad (7.18)$$

we have

$$\begin{aligned} I_1 &\ll \int_0^{\frac{A}{2}} \frac{e^{-4\pi y(B_1 + \frac{u}{2\sqrt{U}} - \sqrt{n})} dy}{|b| + 1} \\ &\ll \frac{1}{(|b| + 1)(B_1 + \frac{u}{2\sqrt{U}} - \sqrt{n})} = \frac{1}{(|b| + 1)|B_1 + \frac{u}{2\sqrt{U}} - \sqrt{n}|}. \end{aligned}$$

In the same way

$$I_3 \ll \frac{1}{(|b| + 1)|B_1 + \frac{u}{2\sqrt{U}} - \sqrt{n}|}.$$

By (7.15)-(7.18),

$$\begin{aligned} I_2 &\ll \frac{1}{|b| + 1} e^{-2\pi A(B_1 + \frac{u}{2\sqrt{U}} - \sqrt{n})} \int_{\frac{\sqrt{U}}{2}}^{(b_0+1/2)\sqrt{U}} e^{-\frac{4\pi x^2}{A^2}} dx \\ &\ll \frac{A}{|b| + 1} e^{-2\pi A(B_1 + \frac{u}{2\sqrt{U}} - \sqrt{n})} \\ &= \frac{1}{(|b| + 1)|B_1 + \frac{u}{2\sqrt{U}} - \sqrt{n}|} (A|B_1 + \frac{u}{2\sqrt{U}} - \sqrt{n}| e^{-2\pi A|B_1 + \frac{u}{2\sqrt{U}} - \sqrt{n}|}) \\ &\ll \frac{1}{(|b| + 1)|B_1 + \frac{u}{2\sqrt{U}} - \sqrt{n}|}. \end{aligned}$$

Hence the integral (7.17) is

$$\int_C \ll \frac{1}{(|b|+1)|B_1 + \frac{u}{2\sqrt{U}} - \sqrt{n}|} \quad (7.19)$$

for $\sqrt{n} \leq B_1 + \frac{u}{2\sqrt{U}}$. If $\sqrt{n} > B_1 + \frac{u}{2\sqrt{U}}$, then moving the path C to C_A^- , we have also (7.19). Hence, by (7.14) we obtain

$$\begin{aligned} J_2(n) &\ll \sum_{u=0,m} \frac{1}{|B_1 + \frac{u}{2\sqrt{U}} - \sqrt{n}|} \sum_b' \frac{1}{(|b|+1)} \\ &\ll L \sum_{u=0,m} \frac{1}{|B_1 + \frac{u}{2\sqrt{U}} - \sqrt{n}|}. \end{aligned} \quad (7.20)$$

In the same way,

$$J_1 \ll \sum_{\nu=0,1} \frac{1}{|B_1 + \frac{\nu}{2\sqrt{U}} - \sqrt{n}|} = \sum_{u=0,m} \frac{1}{|B_1 + \frac{u}{2\sqrt{U}} - \sqrt{n}|}. \quad (7.21)$$

Therefore, it follows from (7.12), (7.20), (7.21) and Lemma 3.1 that

$$\begin{aligned} Q_{11}(B_1) + Q_{12}(B_1) &\ll \sum_{u=0,m} \sum_{-L^2 < \sqrt{n} - \sqrt{U} \leq L^2} \frac{Ld(n)n^{-1/4}}{|B_1 + \frac{u}{2\sqrt{U}} - \sqrt{n}|} \\ &\ll U^{\frac{1}{4}+\varepsilon} \sum_{u=0,m} \sum_{-L^2\sqrt{U} < n-U \leq L^2\sqrt{U}} \frac{1}{|n - (B_1 + \frac{u}{2\sqrt{U}})^2|} \\ &\ll U^{\frac{1}{4}+\varepsilon} (1 + \sum_{u=0,m} (\frac{1}{\{(B_1 + \frac{u}{2\sqrt{U}})^2\}} + \frac{1}{1 - \{(B_1 + \frac{u}{2\sqrt{U}})^2\}})), \end{aligned} \quad (7.22)$$

where $\varepsilon = O((\log L)^{-1})$. Moreover,

$$\begin{aligned} \{(B_1 + \frac{u}{2\sqrt{U}})^2\} &= \{(\sqrt{U} + \frac{j+u}{2\sqrt{U}} + \eta + \xi)^2\} \\ &= \{U + j + u + 2\sqrt{U}(\eta + \xi) + (\frac{j+u}{2\sqrt{U}} + \eta + \xi)^2\} \\ &= \{2\sqrt{U}\xi + O(L^{-1})\} \end{aligned}$$

for $u = 0, m$, $2\sqrt{U}\eta = 2k_1$, $j, u \ll \sqrt{U}L^{-2}$, $\eta \ll L^{-1}$. Hence by (1.18): $\frac{1}{8} \leq 2\sqrt{U}\xi \leq \frac{3}{8}$, we have

$$\frac{1}{9} \leq \{(B_1 + \frac{u}{2\sqrt{U}})^2\} \leq \frac{4}{9}. \quad (7.23)$$

Furthermore, by (7.22) and (7.23), we have

$$Q_{11}(B_1) + Q_{12}(B_1) = \delta_{Q_{11}} \ll U^{\frac{1}{4}+\varepsilon}. \quad (7.24)$$

This and (7.11) give

$$\begin{aligned}
Q_1(B_1) &= Q_{10}(B_1) + \delta_{Q_{11}} \\
&= \sqrt{2} \sum_{-L^2 < \sqrt{n} - \sqrt{U} \leq L^2} d(n)n^{-1/4} J_0(n) + \delta_{Q_{11}} \\
&= \sqrt{2} \sum_{b=1}^{b_0} \int_0^m du \sum_{-L^2 < \sqrt{n} - \sqrt{U} \leq L^2} d(n)n^{-1/4} \int_C f_{nb}(z, u) dz + \delta_{Q_{11}}, \quad (7.25)
\end{aligned}$$

where C is from $\frac{\sqrt{U}}{2}$ to $(b_0 + \frac{1}{2})\sqrt{U}$, $\delta_{Q_{11}} \ll U^{\frac{1}{4}+\varepsilon}$ and

$$f_{nb}(z, u) = e^{-\frac{4\pi z^2}{A^2}} e(2(B_1 - \sqrt{n})z + (\frac{z}{\sqrt{U}} - b)u). \quad (7.26)$$

Clearly,

$$\begin{aligned}
\int_0^m du \sum_{-L^2 < \sqrt{n} - \sqrt{U} \leq L^2} &= \int_0^m du \left(\sum_{\sqrt{U} - L^2 < \sqrt{n} \leq B_1 + \frac{u}{2\sqrt{U}}} + \sum_{B_1 + \frac{u}{2\sqrt{U}} < \sqrt{n} \leq \sqrt{U} + L^2} \right) \\
&= \sum_{\sqrt{U} - L^2 < \sqrt{n} \leq B_1 + \frac{m}{2\sqrt{U}}} \int_{M_1(n)}^m + \sum_{B_1 < \sqrt{n} \leq \sqrt{U} + L^2} \int_0^{M_2(n)}, \quad (7.27)
\end{aligned}$$

where

$$M_1(n) = \max(0, 2\sqrt{U}(\sqrt{n} - B_1)), \quad (7.28)$$

$$M_2(n) = \min(m, 2\sqrt{U}(\sqrt{n} - B_1)). \quad (7.29)$$

Using

$$\int_C f_{nb}(z, u) dz = \int_{C_A^+} = \int_{C_A^-},$$

where C_A^\pm be as Fig 7.1, by (7.25) and (7.27) we have

$$Q_1(B_1) = \sqrt{2} \sum_{b=1}^{b_0} (w^+(b) + w^-(b)) + \delta_{Q_{11}}, \quad (7.30)$$

where

$$\delta_{Q_{11}} = O(U^{\frac{1}{4}+\varepsilon}), \quad \varepsilon = O((\log L)^{-1}),$$

$$\begin{aligned}
w^+(b) &= \sum_{\sqrt{U} - L^2 < \sqrt{n} \leq B_1 + \frac{m}{2\sqrt{U}}} d(n)n^{-1/4} \int_{C_A^+} dz \int_{M_1(n)}^m f_{nb}(z, u) du \\
&= \sum_{\sqrt{U} - L^2 < \sqrt{n} \leq B_1 + \frac{m}{2\sqrt{U}}} d(n)n^{-1/4} \int_{C_A^+} \frac{e^{-\frac{4\pi z^2}{A^2}} e(2(B_1 + \frac{u}{2\sqrt{U}} - \sqrt{n})z - bu)}{2\pi i(\frac{z}{\sqrt{U}} - b)} \Big|_{u=M_1(n)}^{u=m}. \quad (7.31)
\end{aligned}$$

In the same way,

$$w^-(b) = \sum_{B_1 < \sqrt{n} \leq \sqrt{U} + L^2} d(n)n^{-1/4} \int_{C_A^-} \frac{e^{-\frac{4\pi z^2}{A^2}} e(2(B_1 + \frac{u}{2\sqrt{U}} - \sqrt{n})z - bu)}{2\pi i(\frac{z}{\sqrt{U}} - b)} \Big|_{u=0}^{u=M_2(n)} \quad (7.32)$$

For a function $g_n(u)$, by (7.28) we have

$$\begin{aligned} & \sum_{\sqrt{U} - L^2 < \sqrt{n} \leq B_1 + \frac{m}{2\sqrt{U}}} g_n(u) \Big|_{u=M_1(n)}^{u=m} \\ &= \sum_{\sqrt{U} - L^2 < \sqrt{n} \leq B_1} g_n(u) \Big|_{u=0}^{u=m} + \sum_{B_1 < \sqrt{n} \leq B_1 + \frac{m}{2\sqrt{U}}} g_n(u) \Big|_{u=2\sqrt{U}(\sqrt{n} - \sqrt{U})}^{u=m} \\ &= \sum_{\sqrt{U} - L^2 < \sqrt{n} \leq B_1 + \frac{u}{2\sqrt{U}}} g_n(u) \Big|_{u=0}^{u=m} - \sum_{0 < \sqrt{n} - B_1 \leq \frac{m}{2\sqrt{U}}} g_n(2\sqrt{U}(\sqrt{n} - B_1)) \quad (7.33) \end{aligned}$$

Likewise ,

$$\begin{aligned} & \sum_{B_1 < \sqrt{n} \leq \sqrt{U} + L^2} g_n(u) \Big|_{u=0}^{u=M_2(n)} = \\ &= \sum_{B_1 + \frac{u}{2\sqrt{U}} < \sqrt{n} \leq \sqrt{U} + L^2} g_n(u) \Big|_{u=0}^{u=m} + \sum_{0 < \sqrt{n} - B_1 \leq \frac{m}{2\sqrt{U}}} g_n(2\sqrt{U}(\sqrt{n} - B_1)) \quad (7.34) \end{aligned}$$

Noticing $e(-ub) = 1$ for $u = 0, m$, and

$$\begin{aligned} & e(2(B_1 + \frac{u}{2\sqrt{U}} - \sqrt{n})z - bu) \Big|_{u=2\sqrt{U}(\sqrt{n} - B_1)} = \\ &= e(-2b\sqrt{U}(\sqrt{n} - B_1)) = e(-2b\sqrt{nU} + 2b\sqrt{U}(\sqrt{U} + \frac{j}{2\sqrt{U}} + \eta + \xi)) \\ &= e(-2b\sqrt{nU} + 2b\sqrt{U}\xi) \end{aligned}$$

for $\eta = k_1/\sqrt{U}$, we know that (7.30)-(7.34) deduce

$$Q_1(B_1) = \sqrt{2} \sum_{b=1}^{b_0} (w_1^+(b) + w_1^-(b) + w_0(b)) + \delta_{Q_{11}}, \quad (7.35)$$

where

$$w_1^+(b) = \sum_{\sqrt{U} - L^2 < \sqrt{n} \leq B_1 + \frac{u}{2\sqrt{U}}} d(n)n^{-1/4} \int_{C_A^+} \frac{e^{-\frac{4\pi z^2}{A^2}} e(2(B_1 + \frac{u}{2\sqrt{U}} - \sqrt{n})z)}{2\pi i(\frac{z}{\sqrt{U}} - b)} dz \Big|_{u=0}^{u=m} \quad (7.36)$$

$$w_1^-(b) = \sum_{B_1 + \frac{u}{2\sqrt{U}} < \sqrt{n} \leq \sqrt{U} + L^2} d(n)n^{-1/4} \int_{C_A^-} \frac{e^{-\frac{4\pi z^2}{A^2}} e(2(B_1 + \frac{u}{2\sqrt{U}} - \sqrt{n})z)}{2\pi i(\frac{z}{\sqrt{U}} - b)} dz \Big|_{u=0}^{u=m} \quad (7.37)$$

$$\begin{aligned} w_0(b) &= \sum_{0 < \sqrt{n} - B_1 \leq \frac{m}{2\sqrt{U}}} d(n)n^{-1/4} \oint_{C_A^- - C_A^+} \frac{e^{-\frac{4\pi z^2}{A^2}} e(-2b\sqrt{nU} + 2b\sqrt{U}\xi) dz}{2\pi i(\frac{z}{\sqrt{U}} - b)} \\ &= \sqrt{U} e^{-\frac{4\pi Ub^2}{A^2}} e(2b\sqrt{U}\xi) \sum_{0 < \sqrt{n} - B_1 \leq \frac{m}{2\sqrt{U}}} d(n)n^{-1/4} e(-2b\sqrt{nU}) \end{aligned} \quad (7.38)$$

Repeating the proof of (7.24),

$$\delta_{Q_{10}} = \sum_{b=1}^{b_0} (w_1^+(b) + w_1^-(b)) \ll U^{\frac{1}{4}+\varepsilon}, \quad \varepsilon = O\left(\frac{1}{\log L}\right), \quad (7.39)$$

we obtain from (7.35)-(7.39) that

$$Q_1(B_1) = \sqrt{2U} \sum_{b=1}^{b_0} e^{-\frac{4\pi Ub^2}{A^2}} e(2b\sqrt{U}\xi) \sum_{0 < \sqrt{n} - B_1 \leq \frac{m}{2\sqrt{U}}} d(n)n^{-1/4} e(-2b\sqrt{nU}) + \delta_{Q_1}, \quad (7.40)$$

where $\delta_{Q_1} = \delta_{Q_{10}} + \delta_{Q_{11}}$. Therefore, using $e((U+n)b) = 1$ and

$$e(-2b\sqrt{nU}) = e((U+n)b - 2b\sqrt{nU}) = e(b(\sqrt{n} - \sqrt{U})^2),$$

we have

$$Q_1(B_1) = \frac{\sqrt{U}}{\sqrt{2}} \sum_{0 < \sqrt{n} - B_1 \leq \frac{m}{2\sqrt{U}}} d(n)n^{-1/4} N(n) + \delta_{Q_{11}}, \quad (7.41)$$

where

$$\begin{aligned} N(n) &= 2 \sum_{b=1}^{b_0} e^{-\frac{4\pi Ub^2}{A^2}} e(b(2\sqrt{U}\xi + (\sqrt{n} - \sqrt{U})^2)) \\ &= 2 \sum_{b=1}^{\infty} e^{-\frac{4\pi Ub^2}{A^2}} e(b(2\sqrt{U}\xi + (\sqrt{n} - \sqrt{U})^2)) + O(U^{-10}), \end{aligned}$$

the last equality is given by

$$e^{-\frac{4\pi Ub^2}{A^2}} = e^{-\frac{4\pi b^2}{C_0^2}} \ll U^{-10} e^{-\frac{b^2}{C_0^2}}$$

for $b > b_0 = [C_0^2 L]$, $C_0 \geq 200$. By Lemma 3.2,

$$\begin{aligned} \text{Re} N(n) &= -1 + \sum_{b=-\infty}^{\infty} e^{-\frac{4\pi Ub^2}{A^2}} e(b(2\sqrt{U}\xi + (\sqrt{n} - \sqrt{U})^2)) + O(U^{-10}) \\ &= -1 + \frac{A}{2\sqrt{U}} \sum_{b=-\infty}^{\infty} e^{-\frac{\pi A^2}{4U} (2\sqrt{U}\xi + (\sqrt{n} - \sqrt{U})^2 - b)^2} + O(U^{-10}). \end{aligned}$$

By (7.41),

$$(\sqrt{n} - \sqrt{U})^2 = O(L^{-1}).$$

Thus,

$$\frac{1}{9} \leq 2\sqrt{U}\xi + (\sqrt{n} - \sqrt{U})^2 \leq \frac{3}{7}$$

for

$$\frac{1}{8} \leq 2\sqrt{U}\xi \leq \frac{3}{8}.$$

Hence

$$\begin{aligned} \operatorname{Re} N(n) &= -1 + O(\sqrt{L}e^{-\frac{\pi C_0^2 L}{4 \cdot 11^2}}) + O(\sqrt{L} \sum_{b=1}^{\infty} e^{-\frac{C_0^2 L b^2}{4}}) + O(U^{-10}) \\ &= -1 + O(U^{-10}). \end{aligned} \quad (7.42)$$

It follows from (7.41) and (7.42) that

$$Q_1(B_1) = -\frac{\sqrt{U}}{\sqrt{2}} \sum_{0 < \sqrt{n} - B_1 \leq \frac{m}{2\sqrt{U}}} d(n)n^{-1/4} + \delta_{Q_1}, \quad (7.43)$$

where $\delta_{Q_1} = \delta_{Q_{10}} + \delta_{Q_{11}} \ll U^{\frac{1}{4}+\varepsilon}$, $0 < \varepsilon \ll \frac{1}{\log L}$. The proof of (7.4) is complete. \square

Note

For other application to be able, let us write down the accurate $\delta_{Q_1}(B_1)$, the O -term of (7.43):

$$\delta_{Q_1}(B_1) = \delta_{Q_{10}} + \delta_{Q_{11}} = Q_{11} + Q_{12} + \sqrt{2} \sum_{b=1}^{b_0} (w_1^+(b) + w_1^-(b)) + O(U^{-\frac{1}{4}+\varepsilon(U)}), \quad (7.44)$$

where $\varepsilon(U) \ll 1/\log \log U$,

$$\begin{aligned} Q_{1j}(B_1) &= \sqrt{2} \sum_{-L^2 < \sqrt{n} - \sqrt{U} \leq L^2} d(n)n^{-1/4} \int_C e^{-\frac{4\pi z^2}{A^2}} e(2(B_1 - \sqrt{n})z) \rho_j(z) dz, \\ j &= 1, 2, \end{aligned} \quad (7.45)$$

$$\rho_1(z) = \frac{1}{2} + \frac{1}{2} e\left(\frac{zm}{\sqrt{U}}\right),$$

$$\rho_2(z) = \sum_b \left. \frac{e\left(\frac{zu}{\sqrt{U}}\right)}{2\pi i \left(\frac{z}{\sqrt{U}} - b\right)} \right|_{u=0}^{u=m},$$

$$w_1^+(b) = \sum_{\sqrt{U}-L^2 < \sqrt{n} \leq B_1 + \frac{u}{2\sqrt{U}}} d(n)n^{-1/4} \int_{C_A^+} f_{nb}(z, u) \Big|_{u=0}^{u=m} dz, \quad (7.46)$$

$$w_1^-(b) = \sum_{B_1 + \frac{u}{2\sqrt{U}} < \sqrt{n} \leq \sqrt{U} + L^2} d(n)n^{-1/4} \int_{C_A^-} f_{nb}(z, u) \Big|_{u=0}^{u=m} dz, \quad (7.47)$$

where C is from $\sqrt{U}/2$ to $(b_0 + 1/2)\sqrt{U}$, \sum' is for $-U^{10} \leq b \leq 0$ or $b_0 + 1 \leq b \leq U^{10}$, C_A^\pm is Fig 7.1,

$$f_{nb}(z, u) = \frac{e^{-\frac{4\pi z^2}{A^2}} e(2(B_1 + \frac{u}{2\sqrt{U}} - \sqrt{n})z)}{2\pi i(\frac{z}{\sqrt{U}} - b)}. \quad (7.48)$$

8 EVALUATION OF SUM $\sum_{k=0}^m T(B + \xi)$

Let

$$\begin{aligned} Q(B_1) &= \sum_{k=0}^m T(B + \xi) = \sum_{k=0}^m (T_0(B_1 + \frac{k}{2\sqrt{U}}) + T_1(B_1 + \frac{k}{2\sqrt{U}})) \\ &= Q_0(B_1) + Q_1(B_1), \quad \text{say} \end{aligned} \quad (8.1)$$

where $T_0(B_1)$ is (5.14), $T_1(B_1)$ is (5.18) and

$$B_1 = \sqrt{U} + \frac{j}{2\sqrt{U}} + \eta + \xi, \quad m \ll \sqrt{U}L^{-2}. \quad (8.2)$$

In this section, we are going to prove that

$$\begin{aligned} \text{Re } Q(B_1) &= 2\sqrt{U} \sum_{\sqrt{n} \leq \frac{\sqrt{U}}{2}} d(n)n^{-\frac{1}{4}} e^{-\frac{4\pi n}{A^2}} \int_0^{\frac{m}{2\sqrt{U}}} \cos(4\pi\sqrt{n}(B_1 + x) + \frac{\pi}{4}) dx \\ &\quad + \text{Re } \delta_Q(B_1), \end{aligned} \quad (8.3)$$

where $\delta_Q \ll U^{\frac{1}{4}+\varepsilon}$.

(This equality is a key of success of our paper. But if there was no sum of the right side, we could replace k and $\sum_{k=0}^m S(B)$ by $k + k'$ and $\sum_{k,k'=0}^m S(B)$, respectively, and we should be succeed, although it would be more numerous.)

Proof of (8.3):

First, we evaluate further $\text{Re } Q_1(B_1)$. By (7.23):

$$\frac{1}{9} \leq \{(B_1 + \frac{u}{2\sqrt{U}})^2\} \leq \frac{4}{9}$$

for $u = 0, m$. For $A = C_0\sqrt{UL}$, $C_0 \geq 200$, $-\frac{5\sqrt{L}}{A} \leq \theta \leq \frac{5\sqrt{L}}{A}$ and $u = 0, m$, we have

$$\begin{aligned} &|(B_1 + \frac{u}{2\sqrt{U}} + \theta)^2 - (B_1 + \frac{u}{2\sqrt{U}})^2| \\ &= |\theta||2B_1 + \frac{u}{\sqrt{U}} + \theta| \leq \frac{5}{200\sqrt{U}}(2\sqrt{U} + O(1)) < \frac{1}{19} \end{aligned} \quad (8.4)$$

Hence it does not run over any integer number from $(B_1 + \frac{u}{2\sqrt{U}})^2$ to $(B_1 + \frac{u}{2\sqrt{U}} + \theta)^2$ when $u = 0, m$. So that

$$\begin{aligned} \int_0^{\frac{5\sqrt{L}}{A}} e^{-\pi A^2 \theta^2} \sum_{0 < \sqrt{n} - B_1 - \frac{u}{2\sqrt{U}} \leq \theta} &= 0, \\ \int_0^{\frac{5\sqrt{L}}{A}} e^{-\pi A^2 \theta^2} \sum_{-\theta \leq \sqrt{n} - B_1 - \frac{u}{2\sqrt{U}} \leq 0} &= 0. \end{aligned}$$

Hence

$$\begin{aligned} A \int_{-\frac{5\sqrt{L}}{A}}^{\frac{5\sqrt{L}}{A}} e^{-\pi A^2 \theta^2} d\theta \sum_{0 < \sqrt{n} - B_1 - \theta \leq \frac{m}{2\sqrt{U}}} d(n) n^{-1/4} \\ = (A \int_{-\frac{5\sqrt{L}}{A}}^{\frac{5\sqrt{L}}{A}} e^{-\pi A^2 \theta^2} d\theta) \sum_{0 < \sqrt{n} - B_1 \leq \frac{m}{2\sqrt{U}}} d(n) n^{-1/4} \\ = (1 + O(U^{-10})) \sum_{0 < \sqrt{n} - B_1 \leq \frac{m}{2\sqrt{U}}} d(n) n^{-1/4}, \end{aligned} \quad (8.5)$$

where the last equality is given by

$$A \int_{-\frac{5\sqrt{L}}{A}}^{\frac{5\sqrt{L}}{A}} e^{-\pi A^2 \theta^2} d\theta = A \int_{-\infty}^{\infty} e^{-\pi A^2 \theta^2} d\theta = 1 + O(U^{-10}).$$

It follows from (7.4) and (8.5) that

$$\begin{aligned} \operatorname{Re} Q_1(B_1) &= -\frac{\sqrt{U}A}{\sqrt{2}} \int_{-\frac{5\sqrt{L}}{A}}^{\frac{5\sqrt{L}}{A}} e^{-\pi A^2 \theta^2} \sum_{0 < \sqrt{n} - B_1 - \theta \leq \frac{m}{2\sqrt{U}}} d(n) n^{-1/4} \\ &\quad + \delta_{Q_1}(B_1) + O(U^{-1/4+\varepsilon(U)}), \end{aligned} \quad (8.6)$$

where $\delta_{Q_1}(B_1) \ll U^{1/4+\varepsilon(U)}$ is (7.43). By Lemma 2.4,

$$\begin{aligned} \operatorname{Re} Q_1(B_1) &= -\frac{\sqrt{U}A}{\sqrt{2}} \sum_{n \leq N} d(n) \int_{-\frac{5\sqrt{L}}{A}}^{\frac{5\sqrt{L}}{A}} e^{-\pi A^2 \theta^2} d\theta \int_{(B_1+\theta)^2}^{(B_1+\frac{m}{2\sqrt{U}}+\theta)^2} x^{-1/4} \Theta(nx) dx \\ &\quad + \&_l + \delta_{Q_1}(B_1) + O(U^{-1/4+\varepsilon(U)}) + O(\varepsilon_1) \\ &= -2\sqrt{U} \sum_{n \leq N} d(n) n^{-1/4} \sum_{\alpha=0}^2 \frac{\gamma_\alpha A}{(4\pi\sqrt{n})^\alpha} \int_{-\frac{5\sqrt{L}}{A}}^{\frac{5\sqrt{L}}{A}} e^{-\pi A^2 \theta^2} j_{n\alpha}(\theta) d\theta \\ &\quad + \&_l + \delta_{Q_1}(B_1) + O(U^{-1/4+\varepsilon(U)}) + O(\varepsilon_1) \end{aligned} \quad (8.7)$$

holds for any $\varepsilon_1 > 0$ as $N \geq N(\varepsilon_1)$, where

$$j_{n\alpha}(\theta) = \int_{B_1+\theta}^{B_1+\frac{m}{2\sqrt{U}}+\theta} \frac{\cos(4\pi\sqrt{n}x + \frac{\pi}{4} + \frac{\alpha\pi}{2})}{x^\alpha} dx, \quad (8.8)$$

$$\&l_l = \&l_l(\eta) = -\sqrt{2U} \int_{-\frac{5\sqrt{L}}{A}}^{\frac{5\sqrt{L}}{A}} e^{-\pi A^2 \theta^2} \int_{B_0+\theta}^{B_0+\frac{m}{2\sqrt{U}}+\theta} \delta_x(\eta) dx, \quad (8.9)$$

where

$$B_0 = B_1 - \eta = \sqrt{U} + \frac{j}{2\sqrt{U}} + \xi,$$

$$\delta_x(\eta) = (x + \eta)^{1/2} (\log(x + \eta)^2 + 2\gamma).$$

By Lemma 3.6,

$$\delta_x(\eta) \ll_\eta U^{-99}$$

(the definition of " \ll_η " see (1.28)), we know that

$$\&l_l = \&l_l(\eta) \ll_\eta U^{-90}. \quad (8.10)$$

Moreover,

$$\begin{aligned} & A \int_{-\frac{5\sqrt{L}}{A}}^{\frac{5\sqrt{L}}{A}} e^{-\pi A^2 \theta^2} j_{n\alpha}(\theta) d\theta \\ &= A \int_{-\frac{5\sqrt{L}}{A}}^{\frac{5\sqrt{L}}{A}} e^{-\pi A^2 \theta^2} d\theta \frac{\sin(4\pi\sqrt{n}(x+\theta) + \frac{\pi}{4} + \frac{\alpha\pi}{2})}{4\pi\sqrt{n}(x+\theta)^\alpha} \Big|_{x=B_1}^{x=B_1+\frac{m}{2\sqrt{U}}} \\ & \quad + A \int_{-\frac{5\sqrt{L}}{A}}^{\frac{5\sqrt{L}}{A}} e^{-\pi A^2 \theta^2} d\theta \int_{B_1}^{B_1+\frac{m}{2\sqrt{U}}} \frac{\alpha \sin(4\pi\sqrt{n}(x+\theta) + \frac{\pi}{4} + \frac{\alpha\pi}{2})}{4\pi\sqrt{n}(x+\theta)^{\alpha+1}} dx \\ &\ll \frac{A}{n(\sqrt{U})^\alpha}, \end{aligned} \quad (8.11)$$

where the last inequality is given by the integration by parts for θ . Hence, using

$$\sum_{U^3 < n \leq N} d(n) n^{-1/4} \frac{A}{n} \ll U^{-1/4} L^2, \quad (8.12)$$

$$\sum_{U/4 < n \leq N} d(n) n^{-1/4-\alpha/2} \frac{A}{n(\sqrt{U})^\alpha} \ll U^{-1/4} \quad (8.13)$$

for $\alpha \geq 1$, by (8.7), (8.10), (8.12), (8.13) and putting $\varepsilon_1 \rightarrow 0_+$, we have

$$\begin{aligned} \operatorname{Re} Q_1(B_1) &= -2\sqrt{U} \sum_{\frac{U}{4} < n \leq U^3} d(n) n^{-1/4} A \int_{-\frac{5\sqrt{L}}{A}}^{\frac{5\sqrt{L}}{A}} e^{-\pi A^2 \theta^2} j_{n0}(\theta) d\theta \\ &\quad + Q_{10}(B_1) + \operatorname{Re} \delta_{Q_1}(B_1) + O_\eta(U^{-1/4}), \end{aligned} \quad (8.14)$$

where

$$\begin{aligned}
Q_{10}(B_1) &= -2\sqrt{U} \sum_{\sqrt{n} \leq \frac{\sqrt{U}}{2}} d(n)n^{-1/4} \sum_{\alpha=0}^2 \frac{\gamma_\alpha A}{(4\pi\sqrt{n})^\alpha} \int_{-\frac{5\sqrt{L}}{A}}^{\frac{5\sqrt{L}}{A}} e^{-\pi A^2 \theta^2} j_{n\alpha}(\theta) d\theta \\
&= -2\sqrt{U} \sum_{\sqrt{n} \leq \frac{\sqrt{U}}{2}} d(n)n^{-1/4} \sum_{\alpha=0}^2 \frac{\gamma_\alpha A}{(4\pi\sqrt{n})^\alpha} \int_{-\frac{5\sqrt{L}}{A}}^{\frac{5\sqrt{L}}{A}} e^{-\pi A^2 \theta^2} d\theta \times \\
&\quad \times \int_{B_1+\theta}^{B_1+\frac{m}{2\sqrt{U}}+\theta} \frac{\cos(4\pi\sqrt{n}x + \frac{\pi}{4} + \frac{\alpha\pi}{2})}{x^\alpha} dx.
\end{aligned} \tag{8.15}$$

Furthermore,

$$\begin{aligned}
A \int_{-\frac{5\sqrt{L}}{A}}^{\frac{5\sqrt{L}}{A}} e^{-\pi A^2 \theta^2} j_{n0}(\theta) d\theta &= \\
&= \int_{B_1}^{B_1+\frac{m}{2\sqrt{U}}} dx \operatorname{Re} A \int_{-\frac{5\sqrt{L}}{A}}^{\frac{5\sqrt{L}}{A}} e^{-\pi A^2 \theta^2} e(2\sqrt{n}(x+\theta) + 1/8) d\theta \\
&= \int_{B_1}^{B_1+\frac{m}{2\sqrt{U}}} dx \operatorname{Re} e(2\sqrt{n}x + 1/8) A \int_{-\infty}^{\infty} e^{-\pi A^2 \theta^2} e(2\sqrt{n}\theta) d\theta + O(U^{-10}) \\
&= e^{-\frac{4\pi n}{A}} \int_{B_1}^{B_1+\frac{m}{2\sqrt{U}}} \cos(4\pi\sqrt{n}x + \pi/4) dx + O(U^{-10}) \\
&= e^{-\frac{4\pi n}{A}} \int_0^{\frac{m}{2\sqrt{U}}} \cos(4\pi\sqrt{n}(B_1+x) + \pi/4) dx + O(U^{-10}).
\end{aligned} \tag{8.16}$$

Noticing $e^{-\frac{4\pi n}{A}} \ll U^{-10}$ for $\sqrt{n} > b_0\sqrt{U}$, it follows from (8.14) and (8.16) that

$$\begin{aligned}
\operatorname{Re} Q_1(B_1) &= -2\sqrt{U} \int_0^{\frac{m}{2\sqrt{U}}} \left(\sum_{\frac{\sqrt{U}}{2} < \sqrt{n} \leq b_0\sqrt{U}} d(n)n^{-\frac{1}{4}} e^{-\frac{4\pi n}{A^2}} \cos(4\pi\sqrt{n}(B_1+x) + \pi/4) \right) dx \\
&\quad + Q_{10}(B_1) + \delta_{Q_1}(B_1) + O_\eta(U^{-1/4+\varepsilon(U)}),
\end{aligned} \tag{8.17}$$

where $Q_{10}(B_1)$ is (8.15), $\delta_{Q_1}(B_1) \ll U^{\frac{1}{4}+\varepsilon}$ is (7.43).

Next we evaluate $\operatorname{Re} Q_0(B_1)$. By (5.14) and (7.2),

$$\begin{aligned}
Q_0(B_1) &= \sum_{k=0}^m T_0(B+\xi) \\
&= e\left(\frac{1}{8}\right) \sum_{\sqrt{n} \leq \frac{\sqrt{U}}{2}} d(n)n^{-1/4} \sum_{\alpha=0}^2 \gamma_\alpha \left(\frac{i}{4\pi\sqrt{n}}\right)^\alpha A \int_{-\frac{5\sqrt{L}}{A}}^{\frac{5\sqrt{L}}{A}} e^{-\pi A^2 \theta^2} \\
&\quad \times e(2\sqrt{n}(B_1+\theta)) \rho(n, \alpha, \theta) d\theta,
\end{aligned} \tag{8.18}$$

where

$$\rho(n, \alpha, \theta) = \sum_{k=0}^m \frac{e(\frac{\sqrt{n}k}{\sqrt{U}})}{(B_1 + \theta + \frac{k}{2\sqrt{U}})^\alpha}. \quad (8.19)$$

By Lemma 3.3,

$$\begin{aligned} \rho(n, \alpha, \theta) &= \int_0^m \frac{e(\frac{\sqrt{n}u}{\sqrt{U}})du}{(B_1 + \theta + \frac{u}{2\sqrt{U}})^\alpha} + \rho_\delta(n, \alpha, \theta) + O(U^{-9}) \\ &= 2\sqrt{U} \int_{B_1+\theta}^{B_1+\frac{m}{2\sqrt{U}}+\theta} \frac{e(2\sqrt{n}(x - B_1 - \theta))dx}{x^\alpha} \\ &\quad + \rho_\delta(n, \alpha, \theta) + O(U^{-9}), \end{aligned} \quad (8.20)$$

where

$$\begin{aligned} \rho_\delta(n, \alpha, \theta) &= \sum_{-U^{10} \leq b \leq U^{10}, b \neq 0} \int_0^m \frac{e((\frac{\sqrt{n}}{\sqrt{U}} - b)u)du}{(B_1 + \theta + \frac{u}{2\sqrt{U}})^\alpha} + \frac{1}{2(B_1 + \theta)^\alpha} \\ &\quad + \frac{e(\frac{\sqrt{n}m}{\sqrt{U}})}{2(B_1 + \theta + \frac{m}{2\sqrt{U}})^\alpha} \\ &= \sum_{-U^{10} \leq b \leq U^{10}, b \neq 0} \frac{e(\frac{\sqrt{n}u}{\sqrt{U}})}{2\pi i (\frac{\sqrt{n}}{\sqrt{U}} - b)(B_1 + \theta + \frac{u}{2\sqrt{U}})^\alpha} \Big|_{u=0}^{u=m} \\ &\quad + \sum_{-U^{10} \leq b \leq U^{10}, b \neq 0} \int_0^m \frac{\frac{\alpha}{2\sqrt{U}} e((\frac{\sqrt{n}}{\sqrt{U}} - b)u)du}{2\pi i (\frac{\sqrt{n}}{\sqrt{U}} - b)(B_1 + \theta + \frac{u}{2\sqrt{U}})^{\alpha+1}} \\ &\quad + \frac{1}{2(B_1 + \theta)^\alpha} + \frac{e(\frac{\sqrt{n}m}{\sqrt{U}})}{2(B_1 + \theta + \frac{m}{2\sqrt{U}})^\alpha}. \end{aligned} \quad (8.21)$$

It is easy to prove that in Lemma 3.5 ,

$$\begin{aligned} \delta_{Q_0}(B_1) &= \int_{-\frac{5L}{A}}^{\frac{5L}{A}} e^{-\pi\theta^2} d\theta \sum_{\sqrt{n} \leq \frac{\sqrt{U}}{2}} d(n)n^{-1/4} \sum_{\alpha=0}^2 \gamma_\alpha(\frac{i}{4\pi\sqrt{n}})^\alpha e(2\sqrt{n}(B_1 + \theta)) \rho_\delta(n, \alpha) \\ &\ll U^{1/4+\varepsilon}. \end{aligned} \quad (8.22)$$

Then by (8.18), (8.20) and (8.22), we obtain that

$$\begin{aligned} Q_0(B_1) &= 2\sqrt{U}e(\frac{1}{8}) \sum_{\sqrt{n} \leq \frac{\sqrt{U}}{2}} d(n)n^{-1/4} \sum_{\alpha=0}^2 \gamma_\alpha(\frac{i}{4\pi\sqrt{n}})^\alpha A \int_{-\frac{5\sqrt{L}}{A}}^{\frac{5\sqrt{L}}{A}} \\ &\quad \times e^{-\pi A^2 \theta^2} \int_{B_1+\theta}^{B_1+\frac{m}{2\sqrt{U}}+\theta} \frac{e(2\sqrt{n}x)}{x^\alpha} dx + \delta_{Q_0}(B_1) + O(U^{-1/4+\varepsilon(U)}), \end{aligned}$$

where $\delta_{Q_0}(B_1) \ll U^{1/4+\varepsilon(U)}$ is (8.22). Hence

$$\operatorname{Re} Q_0(B_1) = -Q_{10}(B_1) + \delta_{Q_0}(B_1) + O(U^{-1/4+\varepsilon(U)}), \quad (8.23)$$

where $Q_{10}(B_1)$ is (8.15), $\delta_{Q_0}(B_1)$ is (8.22). It follows from (8.17) and (8.23) that

$$\begin{aligned} \operatorname{Re} Q(B_1) &= \operatorname{Re} (Q_0(B_1) + Q_1(B_1)) \\ &= -2\sqrt{U} \int_0^{\frac{m}{2\sqrt{U}}} \left(\sum_{\frac{\sqrt{U}}{2} < \sqrt{n} \leq b_0 \sqrt{U}} d(n) n^{-\frac{1}{4}} e^{-\frac{4\pi n}{A^2}} \right. \\ &\quad \times \cos(4\pi\sqrt{n}(B_1 + x) + \pi/4) dx + \delta_Q(B_1) + O_\eta(U^{-1/4+\varepsilon(U)}). \end{aligned} \quad (8.24)$$

Taking $\nu = \frac{j}{2\sqrt{U}} + x$ in Lemma 4.3, we have $\nu = O(L^{-1})$. By (8.24) and Lemma 4.3 we obtain

$$\begin{aligned} \operatorname{Re} Q(B_1) &= \\ &= 2\sqrt{U} \int_0^{\frac{m}{2\sqrt{U}}} \sum_{\sqrt{n} \leq \frac{\sqrt{U}}{2}} d(n) n^{-\frac{1}{4}} e^{-\frac{4\pi n}{A^2}} \sum_{\alpha=0}^2 \frac{\gamma_\alpha}{(4\pi)^\alpha} \frac{\cos(4\pi\sqrt{n}(B_1 + x) + \pi/4 + \alpha\pi/2) dx}{(\sqrt{n}(\sqrt{n} + B_1 + x))^\alpha} \\ &\quad + \operatorname{Re} \delta_Q(B_1) + O_\eta(U^{-1/4+\varepsilon(U)}) \\ &= 2\sqrt{U} \sum_{\sqrt{n} \leq \frac{\sqrt{U}}{2}} d(n) n^{-\frac{1}{4}} e^{-\frac{4\pi n}{A^2}} \int_0^{\frac{m}{2\sqrt{U}}} \cos(4\pi\sqrt{n}(B_1 + x) + \pi/4) dx \\ &\quad + \operatorname{Re} \delta_q + \operatorname{Re} \delta_Q(B_1) + O_\eta(U^{-1/4+\varepsilon(U)}), \end{aligned} \quad (8.25)$$

where B_1 is (8.2) and

$$\begin{aligned} \delta_Q(B_1) &= \delta_{Q_0}(B_1) + \delta_{Q_1}(B_1) \ll U^{1/4+\varepsilon}, \\ \delta_q &= 2\sqrt{U} \sum_{\sqrt{n} \leq \frac{\sqrt{U}}{2}} d(n) n^{-\frac{1}{4}} \sum_{\alpha=1}^2 \frac{\gamma_\alpha}{(4\pi\sqrt{n})^\alpha} A \int_{-\frac{5\sqrt{U}}{A}}^{\frac{5\sqrt{U}}{A}} e^{-\pi A^2 \theta^2} j_{\alpha n}(\theta) d\theta, \quad (8.26) \\ j_{\alpha n}(\theta) &= \int_\theta^{\frac{m}{2\sqrt{U}} + \theta} \frac{\cos(4\pi\sqrt{n}(B_1 + x) + \pi/4 + \alpha\pi/2) dx}{(\sqrt{n} + B_1 + x)^\alpha} \\ &\ll \frac{1}{\sqrt{nU}}, \text{ for } \alpha \geq 1. \end{aligned}$$

Hence

$$\delta_q \ll \sum_{\sqrt{n} \leq \frac{\sqrt{U}}{2}} d(n) n^{-5/4} = O(1). \quad (8.27)$$

By (8.25) and (8.26) we obtain (8.3) and the proof of (8.3) is complete.

Note

Like §7, we need to write down the accurate $\delta_Q(B_1)$:

$$\delta_Q(B_1) = \delta_{Q_1}(B_1) + \operatorname{Re} \delta_{Q_0}(B_1) + \delta_q(B_1), \quad (8.28)$$

where $\delta_{Q_1}(B_1)$ is (7.44), $\delta_q(B_1)$ is (8.26), $\delta_{Q_0}(B_1)$ is (8.22), i.e.

$$\begin{aligned}
\delta_q(B_1) &= 2\sqrt{U} \sum_{\sqrt{n} \leq \frac{\sqrt{U}}{2}} d(n) n^{-\frac{1}{4}} \sum_{\alpha=1}^2 \frac{\gamma_\alpha}{(4\pi\sqrt{n})^\alpha} A \int_{-\frac{5\sqrt{L}}{A}}^{\frac{5\sqrt{L}}{A}} e^{-\pi A^2 \theta^2} \\
&\quad \times \int_{\theta}^{\frac{m}{2\sqrt{U}} + \theta} \frac{\cos(4\pi\sqrt{n}(B_1 + x) + \pi/4 + \alpha\pi/2) dx}{(\sqrt{n} + B_1 + x)^\alpha} \\
&= \frac{\sqrt{U}}{2\pi} \sum_{\sqrt{n} \leq \frac{\sqrt{U}}{2}} d(n) n^{-\frac{3}{4}} e^{-\frac{4\pi n}{A^2}} \\
&\quad \times \sum_{\alpha=1}^2 \frac{\gamma_\alpha \cos(4\pi\sqrt{n}(B_1 + u) - \pi/4 + \alpha\pi/2)}{(4\pi\sqrt{n}(\sqrt{n} + B_1 + u))^\alpha} \Big|_{u=0}^{u=\frac{m}{2\sqrt{U}}} \\
&\quad + O(U^{-1/4+\varepsilon(U)}), \tag{8.29}
\end{aligned}$$

$$\begin{aligned}
\delta_{Q_0}(B_1) &= \sum_{\sqrt{n} \leq \frac{\sqrt{U}}{2}} d(n) n^{-\frac{1}{4}} \sum_{\alpha=0}^2 \gamma_\alpha \left(\frac{i}{4\pi\sqrt{n}}\right)^\alpha A \int_{-\frac{5\sqrt{L}}{A}}^{\frac{5\sqrt{L}}{A}} e^{-\pi A^2 \theta^2} d\theta \\
&\quad \times e(2\sqrt{n}(B_1 + \theta) + 1/8) \rho_\delta(n, \alpha, \theta), \tag{8.30}
\end{aligned}$$

moreover,

$$\begin{aligned}
\rho_\delta(n, \alpha, \theta) &= \sum_{-U^{10} \leq b \leq U^{10}, b \neq 0} \frac{e(\frac{\sqrt{n}u}{\sqrt{U}})}{2\pi i (\frac{\sqrt{n}}{\sqrt{U}} - b)(B_1 + \theta + \frac{u}{2\sqrt{U}})^\alpha} \Big|_{u=0}^{u=m} \\
&\quad + \sum_{-U^{10} \leq b \leq U^{10}, b \neq 0} \int_0^m \frac{\alpha e((\frac{\sqrt{n}}{\sqrt{U}} - b)u) du}{4\pi i \sqrt{U} (\frac{\sqrt{n}}{\sqrt{U}} - b)(B_1 + \theta + \frac{u}{2\sqrt{U}})^{\alpha+1}} \\
&\quad + \frac{1}{2(B_1 + \theta)^\alpha} + \frac{e(\frac{\sqrt{n}m}{\sqrt{U}})}{2(B_1 + \theta + \frac{m}{2\sqrt{U}})^\alpha} \\
&= \sum_{-U^{10} \leq b \leq U^{10}, b \neq 0} \frac{e(\frac{\sqrt{n}u}{\sqrt{U}})}{2\pi i (\frac{\sqrt{n}}{\sqrt{U}} - b)(B_1 + \frac{u}{2\sqrt{U}})^\alpha} \Big|_{u=0}^{u=m} \\
&\quad + \frac{1}{2(B_1)^\alpha} + \frac{e(\frac{\sqrt{n}m}{\sqrt{U}})}{2(B_1 + \frac{m}{2\sqrt{U}})^\alpha} + \rho_\delta^*(n, \alpha, \theta), \tag{8.31}
\end{aligned}$$

and

$$\begin{aligned}
\rho_\delta^*(n, \alpha, \theta) &= \\
&= \sum_{-U^{10} \leq b \leq U^{10}, b \neq 0} \frac{e(\frac{\sqrt{n}u}{\sqrt{U}})}{2\pi i(\frac{\sqrt{n}}{\sqrt{U}} - b)} \left(\frac{1}{(B_1 + \theta + \frac{u}{2\sqrt{U}})^\alpha} - \frac{1}{(B_1 + \frac{u}{2\sqrt{U}})^\alpha} \right) \Big|_{u=0}^{u=m} \\
&\quad + \frac{1}{2} \left(\frac{1}{(B_1 + \theta)^\alpha} - \frac{1}{B_1^\alpha} \right) + \frac{e(\frac{\sqrt{n}u}{\sqrt{U}})}{2} \left(\frac{1}{(B_1 + \theta + \frac{u}{2\sqrt{U}})^\alpha} - \frac{1}{(B_1 + \frac{u}{2\sqrt{U}})^\alpha} \right) \\
&\quad + \sum_{-U^{10} \leq b \leq U^{10}, b \neq 0} \int_0^m \frac{\alpha e((\frac{\sqrt{n}}{\sqrt{U}} - b)u) du}{4\pi i \sqrt{U} (\frac{\sqrt{n}}{\sqrt{U}} - b) (B_1 + \theta + \frac{u}{2\sqrt{U}})^{\alpha+1}}.
\end{aligned}$$

By Lemma 3.5,

$$\sum_{\sqrt{n} \leq \frac{\sqrt{U}}{2}} d(n) n^{-\frac{1}{4}} \sum_{\alpha=0}^2 \gamma_\alpha \left(\frac{i}{4\pi\sqrt{n}} \right)^\alpha e(2\sqrt{n}(B_1 + \theta) + 1/8) \rho_\delta^*(n, \alpha, \theta) \ll U^{-1/4+\varepsilon(U)}. \quad (8.32)$$

Clearly, as $\alpha = 1, 2$, the sum (8.30) is $O(U^{-\frac{1}{4}+\varepsilon(U)})$. Using

$$A \int_{-\frac{5L}{A}}^{\frac{5L}{A}} e^{-\pi A^2 \theta^2} e(2\sqrt{n}\theta) d\theta = e^{-\frac{4\pi n}{A^2}} + O(U^{-10}),$$

it follows from (8.30), (8.31) and (8.32) that

$$\begin{aligned}
&\text{Re } \delta_{Q_0}(B_1) = \\
&= \sum_{\sqrt{n} \leq \frac{\sqrt{U}}{2}} d(n) n^{-\frac{1}{4}} e^{-\frac{4\pi n}{A^2}} \left(\sum_{-U^{10} \leq b \leq U^{10}, b \neq 0} \frac{\cos(4\pi\sqrt{n}(B_1 + \frac{u}{2\sqrt{U}}) - \frac{\pi}{4})}{2\pi(\frac{\sqrt{n}}{\sqrt{U}} - b)} \Big|_{u=0}^{u=m} \right. \\
&\quad \left. + \frac{\cos(4\pi\sqrt{n}B_1 + \frac{\pi}{4})}{2} + \frac{\cos(4\pi\sqrt{n}(B_1 + \frac{m}{2\sqrt{U}}) + \frac{\pi}{4})}{2} \right) + O(U^{-\frac{1}{4}+\varepsilon(U)}) \\
&= \sum_{\sqrt{n} \leq \frac{\sqrt{U}}{2}} d(n) n^{-\frac{1}{4}} e^{-\frac{4\pi n}{A^2}} \left(\left(\frac{\cos \pi \frac{\sqrt{n}}{\sqrt{U}}}{2 \sin \pi \frac{\sqrt{n}}{\sqrt{U}}} - \frac{\sqrt{U}}{2\pi\sqrt{n}} \right) \cos(4\pi\sqrt{n}(B_1 + \frac{u}{2\sqrt{U}}) - \frac{\pi}{4}) \Big|_{u=0}^{u=m} \right. \\
&\quad \left. + \frac{\cos(4\pi\sqrt{n}B_1 + \frac{\pi}{4})}{2} + \frac{\cos(4\pi\sqrt{n}(B_1 + \frac{m}{2\sqrt{U}}) + \frac{\pi}{4})}{2} \right) + O(U^{-\frac{1}{4}+\varepsilon(U)}), \quad (8.33)
\end{aligned}$$

where the last equality is given by the following (9.25).

9 EVALUATION OF Ω

By (5.13), (8.1) and (8.3),

$$\begin{aligned}
\sum_{k=0}^m S(B) &= 2\sqrt{2}\text{Re} \int_{\xi_1}^{\xi_2} \sum_{k=0}^m T(B + \xi) d\xi + O(U^{-1}) \\
&= 2\sqrt{2} \int_{\xi_1}^{\xi_2} \text{Re } Q(B_1) d\xi + O(U^{-1}) \\
&= 4\sqrt{2U} \int_{\xi_1}^{\xi_2} d\xi \sum_{\sqrt{n} \leq \frac{\sqrt{U}}{2}} d(n) n^{-1/4} e^{-\frac{4\pi n}{A^2}} \int_0^{\frac{m}{2\sqrt{U}}} \cos(4\pi\sqrt{n}(B_1 + x) + \pi/4) dx \\
&\quad + \text{Re} \int_{\xi_1}^{\xi_2} \delta_Q(B_1) d\xi + O_\eta(U^{-3/4+\varepsilon(U)}), \tag{9.1}
\end{aligned}$$

where $0 < \varepsilon(U) \ll 1/\log \log U$, $\delta_Q(B_1) \ll U^{1/4+\varepsilon(U)}$, $\xi_1 = 1/16\sqrt{U}$, $\xi_2 = 3/16\sqrt{U}$ and

$$B_1 = B + \xi - \frac{k}{2\sqrt{U}} = \sqrt{U} + \frac{j}{2\sqrt{U}} + \eta + \xi.$$

Let

$$R(\eta, m) = \frac{1}{\sqrt{V}} \sum_{-\sqrt{V}L \leq j \leq \sqrt{V}L} e^{-\frac{\pi j^2}{V}} \sum_{k=0}^m S(B), \tag{9.2}$$

then by (9.1),

$$\begin{aligned}
R(\eta, m) &= 4\sqrt{2U} \sum_{\sqrt{n} \leq \frac{\sqrt{U}}{2}} d(n) n^{-1/4} e^{-\frac{4\pi n}{A^2}} \int_{\xi_1}^{\xi_2} d\xi \int_0^{\frac{m}{2\sqrt{U}}} v(\eta, \xi, x) dx \\
&\quad + \text{Re } \delta_R(\eta, m) + O_\eta(U^{-3/4+\varepsilon(U)}), \tag{9.3}
\end{aligned}$$

where

$$\delta_R(\eta, m) \ll U^{-1/4+\varepsilon} \tag{9.4}$$

or accurately,

$$\delta_R(\eta, m) = \frac{1}{\sqrt{V}} \sum_{-\sqrt{V}L \leq j \leq \sqrt{V}L} e^{-\frac{\pi j^2}{V}} \int_{\xi_1}^{\xi_2} \delta_Q(B_1) d\xi \tag{9.5}$$

((8.2) and (9.5) deduce (9.4)), moreover,

$$\begin{aligned}
v(\eta, \xi, x) &= \frac{1}{\sqrt{V}} \sum_{-\sqrt{V}L \leq j \leq \sqrt{V}L} e^{-\frac{\pi j^2}{V}} \cos(4\pi n(\sqrt{U} + \frac{j}{2\sqrt{U}} + \xi + \eta + x) + \pi/4) \\
&= \text{Re } e(2\sqrt{n}(\sqrt{U} + \xi + \eta + x) + 1/8) \frac{1}{\sqrt{V}} \sum_{j=-\infty}^{\infty} e^{-\frac{\pi j^2}{V}} e(\frac{\sqrt{n}j}{\sqrt{U}}) + O(U^{-10}).
\end{aligned}$$

By Lemma 3.2,

$$\begin{aligned}
v(\eta, \xi, x) &= \operatorname{Re} e(2\sqrt{n}(\sqrt{U} + \xi + \eta + x) + 1/8) \sum_{j=-\infty}^{\infty} e^{-\pi V(\frac{\sqrt{n}}{\sqrt{U}} - j)^2} + O(U^{-10}) \\
&= e^{-\frac{\pi V n}{U}} \cos(4\pi\sqrt{n}(\sqrt{U} + \xi + \eta + x) + \pi/4) + O\left(\sum_{j=-\infty, j \neq 0}^{\infty} e^{-\pi v(\frac{\sqrt{n}}{\sqrt{U}} - j)^2}\right) + O(U^{-10}) \\
&= e^{-\frac{\pi V n}{U}} \cos(4\pi\sqrt{n}(\sqrt{U} + \xi + \eta + x) + \pi/4) + O(U^{-10})
\end{aligned} \tag{9.6}$$

for $\frac{\sqrt{n}}{\sqrt{U}} \leq 1/2$. Since

$$e^{-\frac{\pi V n}{U}} \ll e^{-L^2}, n > \frac{UL^2}{V},$$

then it follows from (9.3) and (9.6) that

$$\begin{aligned}
R(\eta, m) &= 4\sqrt{2U} \sum_{n \leq \frac{UL^2}{V}} d(n) n^{-1/4} e^{-\pi n(\frac{V}{U} + \frac{4}{A^2})} \\
&\quad \times \int_{\xi_1}^{\xi_2} d\xi \int_0^{\frac{m}{2\sqrt{U}}} \cos(4\pi\sqrt{n}(\sqrt{U} + \xi + \eta + x) + \pi/4) dx \\
&\quad + \delta_R(\eta, m) + O(U^{-3/4+\varepsilon}) \\
&= \frac{\sqrt{2U}}{\pi} \sum_{n \leq \frac{UL^2}{V}} d(n) n^{-3/4} e^{-\pi n(\frac{V}{U} + \frac{4}{A^2})} \\
&\quad \times \int_{\xi_1}^{\xi_2} \cos(4\pi\sqrt{n}(\sqrt{U} + \xi + \eta + x) - \pi/4) dx \Big|_{x=0}^{x=\frac{m}{2\sqrt{U}}} \\
&\quad + \delta_R(\eta, m) + O_\eta(U^{-3/4+\varepsilon(U)}).
\end{aligned} \tag{9.7}$$

It is obvious that

$$\begin{aligned}
f(\eta) \Big|_{\eta_K} &= f(\eta_1 + \dots + \eta_K) \Big|_{\eta_1=0}^{\eta_1=\frac{k_1}{\sqrt{U}}} \cdots \Big|_{\eta_K=0}^{\eta_K=\frac{k_1}{\sqrt{U}}} \\
&= \sum_{a=0}^K (-1)^{K-a} \left(\frac{K}{a}\right) f\left(\frac{ak}{\sqrt{U}}\right) \\
&= (-1)^K f(0) + \sum_{a=1}^K (-1)^{K-a} \left(\frac{K}{a}\right) f\left(\frac{ak_1}{\sqrt{U}}\right).
\end{aligned} \tag{9.8}$$

By (9.7), (9.8) and definition (1.28) of O_η we have

$$R(\eta, m) \Big|_{\eta_K} = R_0 + R_m + R_{mk} + \delta_R(\eta, m) \Big|_{\eta_K} + O(U^{-3/4+\varepsilon(U)}), \tag{9.9}$$

where $0 < \varepsilon(U) \ll 1/\log \log U$,

$$\begin{aligned} R_0 &= (-1)^{K+1} \frac{\sqrt{2U}}{\pi} \sum_{n \leq \frac{UL^2}{V}} d(n) n^{-3/4} e^{-\pi n(\frac{V}{U} + \frac{4}{A^2})} \\ &\quad \times \int_{\xi_1}^{\xi_2} \cos(4\pi \sqrt{n}(\sqrt{U} + \xi) - \pi/4) d\xi, \end{aligned} \quad (9.10)$$

$$\begin{aligned} R_m &= \frac{(-1)^K \sqrt{2U}}{\pi} \sum_{n \leq \frac{UL^2}{V}} d(n) n^{-3/4} e^{-\pi n(\frac{V}{U} + \frac{4}{A^2})} \\ &\quad \times \int_{\xi_1}^{\xi_2} \cos(4\pi \sqrt{n}(\sqrt{U} + \xi + \frac{m}{2\sqrt{U}}) - \pi/4) d\xi, \end{aligned} \quad (9.11)$$

$$\begin{aligned} R_{mk} &= \frac{\sqrt{2U}}{\pi} \sum_{n \leq \frac{UL^2}{V}} d(n) n^{-3/4} e^{-\pi n(\frac{V}{U} + \frac{4}{A^2})} \int_{\xi_1}^{\xi_2} d\xi \sum_{a=1}^K (-1)^{K-a} \left(\frac{K}{a} \right) \\ &\quad \times \cos(4\pi \sqrt{n}(\sqrt{U} + \xi + x + \frac{ak}{\sqrt{U}}) - \pi/4) \Big|_{x=0}^{x=\frac{m}{2\sqrt{U}}}. \end{aligned} \quad (9.12)$$

Therefore,

$$\begin{aligned} \Omega &= \frac{1}{\sqrt{V}} \sum_{-\sqrt{V}L \leq j \leq \sqrt{V}L} e^{-\frac{\pi j^2}{V}} \sum_{k,m=m_0}^{2m_0} S(B) \Big|_{\eta_K} = \sum_{k,m=m_0}^{2m_0} R(\eta, m) \Big|_{\eta_K} \\ &= \Omega_0 + \Omega_1 + \Omega_2 + \Omega_\delta + O(U^{1/4+\varepsilon(U)}), \end{aligned} \quad (9.13)$$

where

$$\Omega_0 = \sum_{k,m=m_0}^{2m_0} R_0 = (m_0 + 1)^2 R_0, \quad (9.14)$$

$$\Omega_1 = \sum_{k,m=m_0}^{2m_0} R_m = (m_0 + 1) \sum_{m=m_0}^{2m_0} R_m, \quad (9.15)$$

$$\Omega_2 = \sum_{k,m=m_0}^{2m_0} R_{mk}, \quad (9.16)$$

$$\Omega_\delta = \sum_{k,m=m_0}^{2m_0} \delta_R(\eta, m) \Big|_{\eta_K} \quad (9.17)$$

(we have replaced k_1 by k), R_0, R_m, R_{mk} and $\delta_R(\eta, m)$ are (9.10), (9.11), (9.12) and (9.5), respectively. By (9.4),

$$\Omega_\delta \ll U^{3/4+\varepsilon(U)}. \quad (9.18)$$

We first evaluate Ω_1 . By (9.11) and (9.15),

$$\Omega_1 = C_1 \sqrt{U} (m_0 + 1) \sum_{\sqrt{n} \leq \frac{UL^2}{V}} d(n) n^{-3/4} e^{-\pi n(\frac{V}{U} + \frac{4}{A^2})} \int_{\xi_1}^{\xi_2} w_{1n}(\xi) d\xi, \quad (9.19)$$

where $C_1 = O(1)$ is real,

$$w_{1n}(\xi) = \sum_{m=m_0}^{2m_0} \cos(4\pi\sqrt{n}(\sqrt{U} + \xi + \frac{m}{2\sqrt{U}}) - \pi/4). \quad (9.20)$$

By Lemma 3.3,

$$\begin{aligned} w_{1n}(\xi) &= \int_{m_0}^{2m_0} \cos(4\pi\sqrt{n}(\sqrt{U} + \xi + \frac{u}{2\sqrt{U}}) - \pi/4) du + w'_\delta + w''_\delta + O(U^{-10}) \\ &= \frac{\sqrt{U}}{2\pi\sqrt{n}} \cos(4\pi\sqrt{n}(\sqrt{U} + \xi + \frac{um_0}{2\sqrt{U}}) - 3\pi/4) \Big|_{u=1}^{u=2} \\ &\quad + w'_\delta + w''_\delta + O(U^{-10}) \end{aligned} \quad (9.21)$$

where

$$w'_\delta = \frac{1}{2} \sum_{b=1}^2 \cos(4\pi\sqrt{n}(\sqrt{U} + \xi + \frac{bm_0}{2\sqrt{U}}) - \pi/4), \quad (9.22)$$

$$\begin{aligned} w''_\delta &= \operatorname{Re} \sum_{-U^{10} \leq b \leq U^{10}, b \neq 0} \int_{m_0}^{2m_0} e(2\sqrt{n}(\sqrt{U} + \xi + \frac{u}{2\sqrt{U}}) - bu - 1/8) du, \\ &= \operatorname{Re} \frac{1}{2\pi i} \sum_{-U^{10} \leq b \leq U^{10}, b \neq 0} \frac{e(2\sqrt{n}(\sqrt{U} + \xi + \frac{u}{2\sqrt{U}}) - 1/8)}{(\frac{\sqrt{n}}{\sqrt{U}} - b)} \Big|_{u=m_0}^{u=2m_0} \end{aligned} \quad (9.23)$$

for $e(-bu) = 1, u = m_0, 2m_0$. It is easy to test Fourier's formula:

$$\cos(\alpha x) = \frac{1}{\pi} \sin(\alpha\pi) \left(\frac{1}{\alpha} + \sum_{b=1}^{\infty} \frac{2\alpha(-1)^b \cos(bx)}{\alpha^2 - b^2} \right), \quad (9.24)$$

$$0 < \alpha < 1, \quad -\pi \leq x \leq \pi.$$

Taking $x = \pi, \alpha = \sqrt{n}/\sqrt{U}$, we have

$$\begin{aligned} \sum_{-U^{10} \leq b \leq U^{10}, b \neq 0} \frac{1}{\frac{\sqrt{n}}{\sqrt{U}} - b} &= \sum_{b=1}^{\infty} \frac{2\frac{\sqrt{n}}{\sqrt{U}}}{(\frac{\sqrt{n}}{\sqrt{U}})^2 - b^2} + O(U^{-10}) \\ &= \frac{\pi \cos(\pi \frac{\sqrt{n}}{\sqrt{U}})}{\sin(\pi \frac{\sqrt{n}}{\sqrt{U}})} - \frac{\sqrt{U}}{\sqrt{n}} \\ &= -\frac{\pi^2 \sqrt{n}}{3\sqrt{U}} (1 + \varphi(\frac{\sqrt{n}}{\sqrt{U}})) + O(U^{-10}), \end{aligned} \quad (9.25)$$

where $\varphi(\frac{\sqrt{n}}{\sqrt{U}}) = C_1 \frac{n}{U} + C_2 (\frac{n}{U})^2 + \dots$ for $\sqrt{n} \leq \frac{UL^2}{V}$. By Lemma 3.5,

$$\sqrt{U}(m_0 + 1) \int_{\xi_1}^{\xi_2} d\xi \sum_{\sqrt{n} \leq \frac{UL^2}{V}} d(n) n^{-3/4} e^{-\pi n(\frac{V}{U} + \frac{4}{A^2})} w''_\delta$$

$$\ll \sqrt{U}(m_0 + 1) \int_{\xi_1}^{\xi_2} U^{-1/2+1/4+\varepsilon(U)} d\xi \ll U^{1/4+\varepsilon(U)}. \quad (9.26)$$

It follows from (9.19), (9.21), (9.22) and (9.26) that

$$\begin{aligned} \Omega_1 &= C_1 U(m_0 + 1) \int_{\xi_1}^{\xi_2} d\xi \sum_{n \leq \frac{UL^2}{V}} d(n) n^{-5/4} e^{-\pi n(\frac{V}{U} + \frac{4}{A^2})} \\ &\quad \times \cos(4\pi\sqrt{n}(\sqrt{U} + \xi + \frac{um_0}{2\sqrt{U}}) - 3\pi/4) \Big|_{u=1}^{u=2} \\ &\quad + \Omega_1 \delta + O(X^{1/4+\varepsilon(X)}), \end{aligned} \quad (9.27)$$

where $U = [X]$,

$$\begin{aligned} \Omega_{1\delta} &= C_2 \sqrt{U}(m_0 + 1) \sum_{b=1}^2 \int_{\xi_1}^{\xi_2} \sum_{n \leq \frac{UL^2}{V}} d(n) n^{-3/4} e^{-\pi n(\frac{V}{U} + \frac{4}{A^2})} \\ &\quad \times \cos(4\pi\sqrt{n}(\sqrt{U} + \xi + \frac{bm_0}{2\sqrt{U}}) - \pi/4) d\xi. \end{aligned} \quad (9.28)$$

Furthermore,

$$e^{-\pi n(\frac{V}{U} + \frac{4}{A^2})} - 1 = -\pi n(\frac{V}{U} + \frac{4}{A^2}) \int_0^1 e^{-\theta \pi n(\frac{V}{U} + \frac{4}{A^2})} d\theta, \quad (9.29)$$

and

$$\begin{aligned} &\cos(4\pi\sqrt{n}(\sqrt{U} + \xi + \frac{um_0}{2\sqrt{U}}) - 3\pi/4) \\ &= \operatorname{Re} e(2\sqrt{n}(\sqrt{X} + \frac{um_0}{2\sqrt{X}}) - 3/8) e(2\sqrt{n}(\xi + \sqrt{U} - \sqrt{X} + \frac{um_0}{2}(\frac{1}{\sqrt{U}} - \frac{1}{\sqrt{X}}))) \\ &= \cos(4\pi\sqrt{n}(\sqrt{X} + \frac{um_0}{2\sqrt{X}}) - 3\pi/4) \\ &\quad - 4\pi\sqrt{n}(\xi + \sqrt{U} - \sqrt{X} + \frac{um_0}{2}(\frac{1}{\sqrt{U}} - \frac{1}{\sqrt{X}})) \cos(4\pi\sqrt{n}(\sqrt{X} + \frac{um_0}{2\sqrt{X}}) - \pi/4) \\ &\quad + \operatorname{Re}(e(2\sqrt{n}(\sqrt{X} + \frac{um_0}{2\sqrt{X}}) - 3/8) \delta_{nu}(\xi)), \end{aligned} \quad (9.30)$$

where

$$\begin{aligned} \delta_{nu}(\xi) &= n(4\pi i(\xi + \sqrt{U} - \sqrt{X} + \frac{um_0}{2}(\frac{1}{\sqrt{U}} - \frac{1}{\sqrt{X}})))^2 \\ &\quad \times \int_0^1 \int_0^1 \theta e(2\theta \theta_1 \sqrt{n}(\xi + \sqrt{U} - \sqrt{X} + \frac{um_0}{2}(\frac{1}{\sqrt{U}} - \frac{1}{\sqrt{X}}))) d\theta d\theta_1. \end{aligned} \quad (9.31)$$

Using (9.29) and Lemma 3.5,

$$U(m_0 + 1) \sum_{n \leq \frac{UL^2}{V}} d(n) n^{-5/4} (e^{-\pi n(\frac{V}{U} + \frac{4}{A^2})} - 1) \int_{\xi_1}^{\xi_2} \cos(4\pi\sqrt{n}(\sqrt{U} + \xi + \frac{um_0}{2\sqrt{U}}) - 3\pi/4) d\xi$$

$$\ll U^{1/4+\varepsilon(U)}. \quad (9.32)$$

Since $U = [X]$, $\xi = O(U^{-\frac{1}{2}})$, then

$$\xi + \sqrt{U} - \sqrt{X} = \xi + \frac{[X] - X}{\sqrt{[X]} + \sqrt{X}} = \xi - \frac{\{X\}}{2\sqrt{X}} + O\left(\frac{1}{X}\right), \quad (9.33)$$

$$\frac{um_0}{2}\left(\frac{1}{\sqrt{U}} - \frac{1}{\sqrt{X}}\right) = O\left(\frac{1}{X}\right), \quad (9.34)$$

for $m_0 \asymp \sqrt{U}L^{-2}$. It follows from (9.31), (9.33), (9.34) and Lemma 3.5 that

$$U(m_0+1) \sum_{n \leq \frac{UL^2}{V}} d(n)n^{-5/4} \int_{\xi_1}^{\xi_2} e(2\sqrt{n}(\sqrt{X} + \frac{um_0}{2\sqrt{X}}) - 3/8) \delta_{nu}(\xi) d\xi \ll X^{1/4+\varepsilon(X)}. \quad (9.35)$$

Therefore, by (9.27), (9.32), (9.33), (9.34) and (9.35) we obtain that

$$\begin{aligned} \Omega_1 &= C_1 U(m_0+1) \sum_{n \leq \frac{UL^2}{V}} d(n)n^{-5/4} \int_{\xi_1}^{\xi_2} (\cos 4\pi\sqrt{n}(\sqrt{X} + \frac{um_0}{2\sqrt{X}}) - 3\pi/4) \\ &\quad + C_2 \sqrt{n}(\xi - \frac{\{X\}}{2\sqrt{X}} + O(\frac{1}{X})) \cos(4\pi\sqrt{n}(\sqrt{X} + \frac{um_0}{2\sqrt{X}}) - \pi/4) d\xi \\ &\quad + \Omega_{1\delta} + O(X^{1/4+\varepsilon(U)}) \\ &= C_0 \sqrt{U}(m_0+1) \sum_{n \leq \frac{UL^2}{V}} d(n)n^{-5/4} \cos(4\pi\sqrt{n}(\sqrt{X} + \frac{um_0}{2\sqrt{X}}) - 3\pi/4) \Big|_{u=1}^{u=2} \\ &\quad + (m_0+1)(C_1 + C_2\{X\}) \sum_{\sqrt{n} \leq \frac{UL^2}{V}} d(n)n^{-3/4} \cos(4\pi\sqrt{n}(\sqrt{X} + \frac{um_0}{2\sqrt{X}}) - \pi/4) \Big|_{u=1}^{u=2} \\ &\quad + \Omega_{1\delta} + O(X^{1/4+\varepsilon(X)}) \end{aligned} \quad (9.36)$$

for $\xi_1 = 1/16\sqrt{U}$, $\xi_2 = 3/16\sqrt{U}$. In the same way, by (9.28),

$$\Omega_{1\delta} = (m_0+1) \sum_{a=1}^2 C_a \sum_{\sqrt{n} \leq \frac{UL^2}{V}} d(n)n^{-3/4} \cos(4\pi\sqrt{n}(\sqrt{X} + \frac{am_0}{2\sqrt{X}}) - \pi/4) + O(X^{1/4+\varepsilon(X)}). \quad (9.37)$$

By Lemma 3.4,

$$\sum_{\frac{UL^2}{V} < n \leq \frac{X_2 L^2}{V}} d(n)n^{-5/4} \cos(4\pi\sqrt{n}(\sqrt{X} + \frac{um_0}{2\sqrt{X}}) - 3\pi/4) \ll X^{-3/4+\varepsilon} \quad (9.38)$$

and

$$\sum_{\frac{UL^2}{V} < n \leq \frac{X_2 L^2}{V}} d(n)n^{-3/4} \cos(4\pi\sqrt{n}(\sqrt{X} + \frac{am_0}{2\sqrt{X}}) - \pi/4) \ll X^{-1/4+\varepsilon}. \quad (9.39)$$

Therefore, by (9.36), (9.37), (9.38) and (9.39), we may rewrite Ω_1 in the form

$$\begin{aligned}
\Omega_1 &= (m_0 + 1)\sqrt{X} \sum_{a=1}^2 C_a \sum_{n \leq \frac{X_2 L^2}{V}} d(n) n^{-5/4} \cos(4\pi\sqrt{n}(\sqrt{X} + \frac{am_0}{2\sqrt{X}}) - 3\pi/4) \\
&\quad + (m_0 + 1) \sum_{a=1}^2 (C_2(a) + C_3(a)\{X\}) \\
&\quad \times \sum_{n \leq \frac{X_2 L^2}{V}} d(n) n^{-3/4} \cos(4\pi\sqrt{n}(\sqrt{X} + \frac{am_0}{2\sqrt{X}}) - \pi/4) \Big|_{u=1}^{u=2} + O(X^{1/4+\varepsilon(U)}) \quad (9.40)
\end{aligned}$$

for $X_1 \leq X \leq X_2$, $U = [X]$, where $C_j(a) = O(1)$ is real and independent of X .

Next we evaluate Ω_2 in (9.16). By (9.16) and (9.12),

$$\begin{aligned}
\Omega_2 &= \frac{\sqrt{2U}}{\pi} \sum_{n \leq \frac{UL^2}{V}} d(n) n^{-3/4} e^{-\pi n(\frac{V}{U} + \frac{4}{A^2})} \\
&\quad \times \sum_{a=1}^K (-1)^K \left(\frac{K}{a}\right) \int_{\xi_1}^{\xi_2} (w_{na}(\xi, \frac{m}{2\sqrt{U}}) - w_{na}(\xi, 0)) d\xi \\
&= \Omega'_2 + \Omega_{20}, \quad \text{say,} \quad (9.41)
\end{aligned}$$

where

$$w_{na}(\xi, \frac{m}{2\sqrt{U}}) = \sum_{k, m=m_0}^{2m_0} \cos(4\pi\sqrt{n}(\sqrt{U} + \xi + \frac{ak}{\sqrt{U}} + \frac{m}{2\sqrt{U}}) - \pi/4) \quad (9.42)$$

$$\begin{aligned}
w_{na}(\xi, 0) &= \sum_{k, m=m_0}^{2m_0} \cos(4\pi\sqrt{n}(\sqrt{U} + \xi + \frac{ak}{\sqrt{U}}) - \pi/4) \\
&= (m_0 + 1) \sum_{k=m_0}^{2m_0} \cos(4\pi\sqrt{n}(\sqrt{U} + \xi + \frac{ak}{\sqrt{U}}) - \pi/4). \quad (9.43)
\end{aligned}$$

Repeating the evaluation of (9.40) we have

$$\begin{aligned}
\Omega_{20} &= -\frac{\sqrt{2U}}{\pi}(m_0 + 1) \sum_{n \leq \frac{UL^2}{V}} d(n)n^{-3/4} e^{-\pi n(\frac{V}{U} + \frac{4}{A^2})} \\
&\quad \times \sum_{a=1}^K (-1)^K \left(\frac{K}{a}\right) \int_{\xi_1}^{\xi_2} \sum_{k=m_0}^{2m_0} \cos(4\pi\sqrt{n}(\sqrt{U} + \xi + \frac{ak}{\sqrt{U}}) - \pi/4) d\xi \\
&= (m_0 + 1)\sqrt{X} \sum_{a=1}^{4K} C_1(a) \sum_{n \leq \frac{X_2 L^2}{V}} d(n)n^{-5/4} \cos(4\pi\sqrt{n}(\sqrt{X} + \frac{am_0}{2\sqrt{X}}) - 3\pi/4) \\
&\quad + (m_0 + 1) \sum_{a=1}^{4K} (C_2(a) + C_3(a)\{X\}) \\
&\quad \times \sum_{n \leq \frac{X_2 L^2}{V}} d(n)n^{-3/4} \cos(4\pi\sqrt{n}(\sqrt{X} + \frac{am_0}{2\sqrt{X}}) - \pi/4) + O(X^{1/4+\varepsilon(U)}), \quad (9.44)
\end{aligned}$$

where $C_j(a)$ is real, and

$$|C_j(a)| \ll \sup_{0 \leq a \leq K} \left(\frac{K}{a}\right) \ll 2^K \ll \exp(O(\frac{L}{\log(L)})). \quad (9.45)$$

Next we evaluate Ω'_2 in (9.41). By (9.42) and Lemma 3.3

$$w_{na}(\xi, \frac{m}{2\sqrt{U}}) = \text{Re} (e(2\sqrt{n}(\sqrt{U} + \xi) - 1/8)\rho_{na}), \quad (9.46)$$

where

$$\rho_{na} = \sum_{k, m=m_0}^{2m_0} e(2\sqrt{n}(\frac{ak}{\sqrt{U}} + \frac{m}{2\sqrt{U}})) = I_1 I_2 + I_1 \sigma_2 + I_2 \sigma_1 + \sigma_1 \sigma_2 + O(U^{-9}), \quad (9.47)$$

moreover,

$$I_1 = \int_{m_0}^{2m_0} e(\frac{2\sqrt{n}au}{\sqrt{U}}) du = \frac{\sqrt{U}}{2\pi i a \sqrt{n}} e(\frac{2\sqrt{n}aum_0}{\sqrt{U}}) \Big|_{u=1}^{u=2}, \quad (9.48)$$

$$I_2 = \int_{m_0}^{2m_0} e(\frac{\sqrt{n}u}{\sqrt{U}}) du = \frac{\sqrt{U}}{2\pi i \sqrt{n}} e(\frac{\sqrt{n}um_0}{\sqrt{U}}) \Big|_{u=1}^{u=2},$$

$$\begin{aligned}
\sigma_1 &= \frac{1}{2} \sum_{b=1}^2 e(\frac{2abm_0}{\sqrt{U}}) + \sum_{-U^{10} \leq b \leq U^{10}, b \neq 0} \frac{e(\frac{2\sqrt{n}am_0u}{\sqrt{U}})}{2\pi i (\frac{2a\sqrt{n}}{\sqrt{U}} - b)} \Big|_{u=1}^{u=2} \\
&= \frac{1}{2} \sum_{b=1}^2 e(\frac{2abm_0}{\sqrt{U}}) - \frac{\pi^2}{3} \frac{e(\frac{2\sqrt{n}am_0u}{\sqrt{U}})}{2\pi i} \frac{\sqrt{n}}{\sqrt{U}} (1 + \varphi(\frac{2a\sqrt{n}}{\sqrt{U}})) \Big|_{u=1}^{u=2} \quad (9.49)
\end{aligned}$$

$$\begin{aligned}
\sigma_2 &= \frac{1}{2} \sum_{b=1}^2 e(\frac{\sqrt{n}bm_0}{\sqrt{U}}) - \frac{\pi^2}{3} \frac{e(\frac{\sqrt{n}bm_0u}{\sqrt{U}})}{2\pi i} \frac{\sqrt{n}}{\sqrt{U}} (1 + \varphi(\frac{\sqrt{n}}{\sqrt{U}})) \Big|_{u=1}^{u=2}. \quad (9.50)
\end{aligned}$$

The last equality is given by (9.25). Since $\sigma_1\sigma_2 \ll 1$, then

$$\begin{aligned} & \sqrt{U} \sum_{\sqrt{n} \leq \frac{UL^2}{V}} d(n)n^{-3/4} e^{-\pi n(\frac{V}{U} + \frac{4}{A^2})} \\ & \quad \times \sum_{a=1}^{K-a} (-1)^{K-a} \left(\frac{K}{a}\right) \sigma_1 \sigma_2 \int_{\xi_1}^{\xi_2} e(2\sqrt{n}(\sqrt{U} + \xi) - 1/8) d\xi \ll X^{1/4+\varepsilon(X)} \end{aligned} \quad (9.51)$$

By (9.41), (9.46), (9.47) and (9.51), running the process of the evaluation of (9.40), we obtain

$$\begin{aligned} \Omega'_2 &= C\sqrt{U} \operatorname{Re} \sum_{n \leq \frac{UL^2}{V}} d(n)n^{-3/4} e^{-\pi n(\frac{V}{U} + \frac{4}{A^2})} \\ & \quad \times \sum_{a=1}^K (-1)^K \left(\frac{K}{a}\right) (I_1 I_2 + I_1 \sigma_2 + I_2 \sigma_1) \int_{\xi_1}^{\xi_2} e(2\sqrt{n}(\sqrt{U} + \xi) - 1/8) d\xi + O(X^{1/4+\varepsilon(X)}). \end{aligned}$$

Thus, we may rewrite in the form:

$$\begin{aligned} \Omega'_2 &= X \sum_{a=1}^{4K+2} C_1(a) \sum_{n \leq \frac{X_2 L^2}{V}} d(n)n^{-7/4} \cos(4\pi\sqrt{n}(\sqrt{X} + \frac{am_0}{2\sqrt{X}}) - \pi/4) \\ & \quad + \sqrt{X} \sum_{a=1}^{4K+2} (C_2(a) + C_3(a)\{X\}) \sum_{n \leq \frac{X_2 L^2}{V}} d(n)n^{-5/4} \cos(4\pi\sqrt{n}(\sqrt{X} + \frac{am_0}{2\sqrt{X}}) - 3\pi/4) \\ & \quad + O(X^{1/4+\varepsilon(X)}), \end{aligned} \quad (9.52)$$

where $C_j(a)$ satisfies (9.45).

Finally, by (9.13), (9.40), (9.44) and (9.52) we obtain that

$$\begin{aligned} \Omega &= \Omega_0 + \Omega_1 + \Omega'_2 + \Omega_{20} + \Omega_\delta + O(X^{1/4+\varepsilon(X)}) \\ &= X \sum_{a=1}^{4K+2} C_1(a) \sum_{n \leq \frac{X_2 L^2}{V}} d(n)n^{-7/4} \cos(4\pi\sqrt{n}(\sqrt{X} + \frac{am_0}{2\sqrt{X}}) - \pi/4) \\ & \quad + \sqrt{X}(m_0 + 1) \sum_{a=1}^{4K} C_2(a) \sum_{n \leq \frac{X_2 L^2}{V}} d(n)n^{-5/4} \cos(4\pi\sqrt{n}(\sqrt{X} + \frac{am_0}{2\sqrt{X}}) - 3\pi/4) \\ & \quad + \sqrt{X} \\ & \quad \times \sum_{a=1}^{4K+2} (C_3(a) + C_4(a)\{X\}) \sum_{n \leq \frac{X_2 L^2}{V}} d(n)n^{-5/4} \cos(4\pi\sqrt{n}(\sqrt{X} + \frac{am_0}{2\sqrt{X}}) - 3\pi/4) \\ & \quad + (m_0 + 1) \\ & \quad \times \sum_{a=1}^{4K} (C_5(a) + C_6(a)\{X\}) \sum_{n \leq \frac{X_2 L^2}{V}} d(n)n^{-3/4} \cos(4\pi\sqrt{n}(\sqrt{X} + \frac{am_0}{2\sqrt{X}}) - \pi/4) \\ & \quad + \Omega_0 + \Omega_\delta + O(X^{1/4+\varepsilon(U)}) \end{aligned} \quad (9.53)$$

for $U = [X]$, $m_0 \asymp \sqrt{X}L^{-2}$, where $C_j(a)$ is real and independent of X ,

$$|C_j(a)| \ll 2^K \ll \exp(O(\frac{L}{\log(L)})), \quad (9.54)$$

Ω_0 is (9.14), Ω_δ is (9.17) and (9.18).

10 PROOF OF THEOREM

By (9.14) and (9.10),

$$\begin{aligned} \Omega_0 &= \frac{(-1)^{K+1} \sqrt{2U} (m_0 + 1)^2}{\pi} \\ &\times \sum_{n \leq \frac{UL^2}{V}} d(n) n^{-3/4} e^{-\pi n(\frac{V}{U} + \frac{4}{A^2})} \int_{\xi_1}^{\xi_2} \cos(4\pi \sqrt{n}(\sqrt{U} + \xi) - \pi/4) d\xi. \end{aligned} \quad (10.1)$$

Since

$$e^{-\pi n(\frac{V}{U} + \frac{4}{A^2})} \ll e^{-L^2}$$

for $\frac{UL^2}{V} < n \leq \frac{X_2 L^2}{V}$, then putting $\xi = \frac{1}{\sqrt{U}} \xi'$, we have

$$\begin{aligned} \Omega_0 &= \frac{2(-1)^{K+1} (m_0 + 1)^2}{\sqrt{2}\pi} \sum_{n \leq \frac{X_2 L^2}{V}} d(n) n^{-3/4} e^{-\pi n(\frac{V}{U} + \frac{4}{A^2})} \\ &\times \int_{1/16}^{3/16} \cos(4\pi \sqrt{n}(\sqrt{U} + \xi/\sqrt{U}) - \pi/4) d\xi + O(U^{-10}) \\ &= \frac{(-1)^{K+1} (m_0 + 1)^2}{4X^{1/4}} \left(\frac{X^{1/4}}{\sqrt{2}\pi} \sum_{n \leq \frac{X_2 L^2}{V}} d(n) n^{-3/4} \cos(4\pi \sqrt{nX} - \pi/4) \right. \\ &\quad \left. + \delta_{01}(X) + \delta_{02}(X) + O(U^{-1}) \right), \end{aligned} \quad (10.2)$$

where

$$\begin{aligned} \delta_{01}(X) &= C_1 X^{1/4} \\ &\times \sum_{n \leq \frac{X_2 L^2}{V}} d(n) n^{-3/4} (e^{-\pi n(\frac{V}{U} + \frac{4}{A^2})} - 1) \int_{1/16}^{3/16} \cos(4\pi \sqrt{n}(\sqrt{U} + \xi/\sqrt{U}) - \pi/4) d\xi, \end{aligned} \quad (10.3)$$

$$\begin{aligned} \delta_{02}(X) &= C_2 X^{1/4} \\ &\times \sum_{n \leq \frac{X_2 L^2}{V}} d(n) n^{-3/4} \int_{1/16}^{3/16} (\cos(4\pi \sqrt{n}(\sqrt{U} + \xi/\sqrt{U}) - \pi/4) - \cos(4\pi \sqrt{nX} - \pi/4)) d\xi, \end{aligned} \quad (10.4)$$

with $C_1, C_2 = O(1)$ be real and independent of X . Denote

$$\delta_{00}(X) = \Delta(X) - \frac{X^{1/4}}{\pi\sqrt{2}} \sum_{n \leq \frac{X_2 L^2}{V}} d(n) n^{-3/4} \cos(4\pi\sqrt{nX} - \pi/4). \quad (10.5)$$

By (10.2),

$$\Omega_0 = \frac{(-1)^{K+1}(m_0 + 1)^2}{4X^{1/4}} (\Delta(X) - \delta_0(X) + O(U^{-1})), \quad (10.6)$$

where

$$\delta_0(X) = \delta_{00}(X) - \delta_{01}(X) - \delta_{02}(X). \quad (10.7)$$

For $X_1 \leq X \leq X_2$, $X_1 \asymp X_2$, $V = X_2^{\varepsilon_0}$, $\varepsilon_0 = 1/\log L$, (2.59) implicates

$$\delta_{00}(X) \ll X^\varepsilon. \quad (10.8)$$

Using (9.29) and taking $\alpha = -1/4$ in Lemma 3.4, we have

$$\delta_{01}(X) \ll X^\varepsilon. \quad (10.9)$$

Since

$$\begin{aligned} & \cos(4\pi\sqrt{n}(\sqrt{U} + \xi/\sqrt{U}) - \pi/4) - \cos(4\pi\sqrt{nX} - \pi/4) = \\ &= \operatorname{Re} e(2\sqrt{nX} - 1/8)(e(2\sqrt{n}(\sqrt{U} + \xi/\sqrt{U}) - \sqrt{X})) - 1) \\ &= \sqrt{n} \operatorname{Re} e(2\sqrt{nX} - 1/8) 4\pi i (\sqrt{U} - \sqrt{X} + \xi/\sqrt{U}) \int_0^1 e(2\theta\sqrt{n}(\sqrt{U} - \sqrt{X} + \xi/\sqrt{U})) d\theta, \end{aligned}$$

then using Lemma 3.4, we have

$$\delta_{02}(X) \ll X^\varepsilon.$$

Thus,

$$\delta_0(X) = \delta_{00}(X) - \delta_{01}(X) - \delta_{02}(X) \ll X^\varepsilon. \quad (10.10)$$

Moreover, by (9.53) and (10.6),

$$\Omega = \frac{(-1)^{K+1}(m_0 + 1)^2}{4X^{1/4}} (\Delta(X) - X^{1/4}\Lambda(X) - \delta(X) + O(X^{-1/2+\varepsilon(X)})), \quad (10.11)$$

where

$$\begin{aligned} \Lambda(X) &= \frac{X}{(m_0 + 1)^2} \sum_{a=1}^{4K+2} C_1(a) \sum_{n \leq \frac{X_2 L^2}{V}} d(n) n^{-7/4} \cos(4\pi\sqrt{n}(\sqrt{X} + \frac{am_0}{2\sqrt{X}}) - \pi/4) \\ &+ \frac{\sqrt{X}}{m_0 + 1} \sum_{a=1}^{4K} C_2(a) \sum_{n \leq \frac{X_2 L^2}{V}} d(n) n^{-5/4} \cos(4\pi\sqrt{n}(\sqrt{X} + \frac{am_0}{2\sqrt{X}}) - 3\pi/4), \end{aligned} \quad (10.12)$$

$$\delta(X) = \delta_0(X) + \delta_1(X) + \delta_2(X), \quad (10.13)$$

The three terms in the right side of the last equality are the following:

$\delta_0(X)$ is (10.10),

$\delta_1(X)$

$$\begin{aligned}
&= \frac{X^{3/4}}{(m_0+1)^2} \sum_{a=1}^{4K+2} (C_3(a) + C_4(a)\{X\}) \sum_{n \leq \frac{X_2 L^2}{V}} d(n) n^{-5/4} \cos(4\pi\sqrt{n}(\sqrt{X} + \frac{am_0}{2\sqrt{X}})) - 3\pi/4) \\
&+ \frac{X^{1/4}}{(m_0+1)} \sum_{a=1}^{4K} (C_5(a) + C_6(a)\{X\}) \sum_{n \leq \frac{X_2 L^2}{V}} d(n) n^{-3/4} \cos(4\pi\sqrt{n}(\sqrt{X} + \frac{am_0}{2\sqrt{X}})) - \pi/4), \\
\delta_2(X) &= \frac{X^{1/4}}{(m_0+1)^2} \Omega_\delta = \frac{X^{1/4}}{(m_0+1)^2} \sum_{k,m=m_0}^{2m_0} \delta_R(\eta, m)|_{\eta_K} \ll X^\varepsilon, \tag{10.14}
\end{aligned}$$

where Ω_δ is (9.17), $\delta_R(\eta, m)$ is (9.5), $C_j(a)$, $\delta_2(X)$ contain new constant $C = O(1)$ to be independent of X .

Therefore, by (1.26) and (10.11) we obtain that

$$\Delta(X) = X^{1/4} \Lambda(X) + \delta(X) + O(X^{-1/2+\varepsilon(X)}), \quad 0 < \varepsilon(X) = O\left(\frac{1}{\log \log X}\right). \tag{10.15}$$

The proof of Theorem is complete. \square

NOTE

For the convenient of the other application afterwards we write down the accurate $\delta(X)$ of (10.15):

$$\begin{aligned}
\delta(X) &= \Delta(X) - \frac{X^{1/4}}{\sqrt{2}\pi} \sum_{n \leq \frac{X_2 L^2}{V}} d(n) n^{-3/4} \cos(4\pi\sqrt{nX} - \pi/4) \\
&+ C_1 X^{1/4} \sum_{n \leq \frac{X_2 L^2}{V}} d(n) n^{-3/4} (e^{-\pi n(\frac{V}{U} + \frac{4}{A^2})} - 1) \int_{1/16}^{3/16} \cos(4\pi\sqrt{n}(\sqrt{U} + \xi/\sqrt{U}) - \pi/4) d\xi \\
&+ C_2 X^{1/4} \sum_{n \leq \frac{X_2 L^2}{V}} d(n) n^{-3/4} \int_{1/16}^{3/16} (\cos(4\pi\sqrt{n}(\sqrt{U} + \xi/\sqrt{U}) - \pi/4) - \cos(4\pi\sqrt{nX} - \pi/4)) d\xi \\
&+ \frac{X^{3/4}}{(m_0+1)^2} \sum_{a=1}^{4K+2} (C_3(a) + C_4(a)\{X\}) \sum_{n \leq \frac{X_2 L^2}{V}} d(n) n^{-5/4} \cos(4\pi\sqrt{n}(\sqrt{X} + \frac{am_0}{2\sqrt{X}})) - 3\pi/4) \\
&+ \frac{X^{1/4}}{(m_0+1)} \sum_{a=1}^{4K} (C_5(a) + C_6(a)\{X\}) \sum_{n \leq \frac{X_2 L^2}{V}} d(n) n^{-3/4} \cos(4\pi\sqrt{n}(\sqrt{X} + \frac{am_0}{2\sqrt{X}})) - \pi/4)
\end{aligned}$$

$$+ \frac{X^{1/4}}{(m_0 + 1)^2 \sqrt{V}} \sum_{-\sqrt{V}L \leq j \leq \sqrt{V}L} e^{-\frac{\pi j^2}{V}} \sum_{k, m=m_0}^{2m_0} \int_{\xi_1}^{\xi_2} \delta_Q(B_1) d\xi|_{\eta_K}, \quad (10.16)$$

where $\delta_Q(B_1)$ is (8.28), $X_1 \leq X \leq X_2$, $U = [X]$, $m_0 \asymp \sqrt{X_2}L^{-2}$, $X_1 \asymp X_2$, $L = \log X_2$, $\xi_1 = 1/16\sqrt{U}$, $\xi_2 = 3/16\sqrt{U}$, $A = C_0\sqrt{UL}$ $C_j(a)$ is independent of X , and

$$|C_j(a)| \ll \exp(O(\frac{L}{\log L})).$$

We can take $V = X_2^{\varepsilon_0}$, $\varepsilon_0 = 1/\log L$, $m_0 = [\sqrt{X_2}L^{-2}]$, such that they are independent of X .

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